Please sign below in acknowledgment and acceptance of the Honor Code.

Signature: __________________________________________

This exam is closed book. You may use the one sheet of formulas/miscellaneous information that you brought. The exam is worth a total of 100 points. The point value of each problem is indicated. Please show all work and clearly mark your answer.

<table>
<thead>
<tr>
<th>Number</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>
1. (14 points) Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let $V$ be a bounded continuous real-valued function on $\Omega$. Consider the following Dirichlet eigenvalue problem.

\[
\begin{cases}
-\Delta u + V(x)u = \lambda u & x \in \Omega \\
u = 0 & x \in \partial \Omega
\end{cases}
\]

(a) Show that the eigenvalues are real.

(b) Show that eigenfunctions corresponding to distinct eigenvalues are orthogonal.
(c) Show that if $V$ is positive, then all the eigenvalues are positive.
2. (8 points) Let \( f(x) = H(x - 1) \sin x \) where

\[
H(x) = \begin{cases} 
1 & x \geq 0 \\
0 & x < 0.
\end{cases}
\]

Define the distribution \( F_f \) associated with \( f \) such that

\[
(F_f, \phi) = \int_{-\infty}^{\infty} f(x) \phi(x) \, dx
\]

for all \( \phi \in \mathcal{D} \). Calculate the distributional derivative of \( F_f \).
3. (10 points) Let \( \Omega = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1, y_1, y_2 > 0\} \). Find the Green’s function for \( \Omega \).
4. (8 points) Consider

\[
\begin{cases}
\Delta u = 0 & x \in \Omega \\
\frac{\partial u}{\partial \nu} + u = g & x \in \partial \Omega.
\end{cases}
\]

(a) State the definition of a single-layer potential with moment \( h \).

(b) In order to write the solution of (*) as a single-layer potential, what equation must \( h \) satisfy?
5. (10 points) Solve

\[
\begin{cases}
    u_t - u_{xx} = 0 & 0 < x < \infty, t > 0 \\
    u(x, 0) = \phi(x) \\
    u(0, t) = g(t)
\end{cases}
\]
6. (10 points) Let $\Omega$ be the triangle with vertices at $(1, 0)$, $(-1, 0)$ and $(0, 2)$. Let $\lambda_1$ be the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial \Omega. \end{cases}$$

Use the Comparison Principle to get an upper bound on the first eigenvalue for this eigenvalue problem. In particular, find the best upper bound on $\lambda_1$ among all rectangles contained within $\Omega$ with sides parallel to the coordinate axes.
7. (10 points) Prove Dirichlet’s principle for Neumann boundary conditions. Let

\[ I(w) = \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx - \int_{\partial \Omega} gw \, dS(x). \]

Let

\[ A = \{ w \in C^2(\Omega) \}. \]

Consider

\[
(*) \begin{cases}
    \Delta u = 0 & x \in \Omega \\
    \frac{\partial u}{\partial n} = g & x \in \partial \Omega
\end{cases}
\]

(a) Show that if \( u \) is a solution of \((*)\), then

\[ I(u) = \min_{w \in A} I(w). \]

(b) Show that if \( I(u) = \min_{w \in A} I(w) \), then \( u \) is a solution of \((*)\).
8. (12 points) Let $\Omega \equiv \{(x,y) \in \mathbb{R}^2 : 0 < x < l, 0 < y < k\}$.

(a) Find all eigenvalues and eigenfunctions for

$$
\begin{cases}
-\Delta X = \lambda X & (x,y) \in \Omega \\
X_y(x,0) = 0, X(x,k) = 0 & 0 < x < l \\
X(0,y) = 0, X_x(l,y) = 0 & 0 < y < k
\end{cases}
$$
(b) Let $X_{nm}(x, y)$ denote the eigenfunctions from part (a). Solve

\[
\begin{aligned}
\begin{cases}
  u_t - \Delta u = 0 & \quad (x, y) \in \Omega, \ t > 0 \\
  u(x, y, 0) = \phi(x, y) & \quad (x, y) \in \Omega \\
  u_y(x, 0, t) = 0, \ u(x, k, t) = 0 & \quad 0 < x < l, \ t > 0 \\
  u(0, y, t) = 0, \ u_x(l, y, t) = 0 & \quad 0 < y < k, \ t > 0 
\end{cases}
\end{aligned}
\]

Express your answer in terms of $X_{nm}(x, y)$. 

9. (10 points) Suppose $u \in C^2(\Omega)$ is a solution of

$$\Delta u = f \geq 0 \quad x \in \Omega.$$ 

Show that

$$u(x) \leq \int_{\partial B(x,r)} u(y) \, dy$$

for all $B(x,r) \subset \Omega$. 

10. (8 points) Find the smooth function \( f \) which yields the best lower bound for \( \int_0^1 (g'(x))^2 \, dx \) among functions satisfying \( g(0) = 3, \ g(1) = 4 \).