Math 220B Final Exam August 16, 2002

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Name: ______________________________

Please sign below in acknowledgment and acceptance of the Honor Code.

Signature: ______________________________

This exam is closed book. You may use the one sheet of formulas/miscellaneous information that you brought. The exam is worth a total of 100 points. The point value of each problem is indicated. Please show all work and clearly mark your answer.

Number	Points
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	
Total	

1. (14 points) Let Ω be an open bounded subset of $\mathbb{R}^n$ and let V be a bounded continuous real-valued function on $\overline{\Omega}$. Consider the following Dirichlet eigenvalue problem.

$$\begin{cases} -\Delta u + V(x)u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

(a) Show that the eigenvalues are real.

(b) Show that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(c) Show that if V is positive, then all the eigenvalues are positive.

2. (8 points) Let $f(x) = H(x-1)\sin x$ where

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Define the distribution F_f associated with f such that

$$(F_f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$$

for all $\phi \in \mathcal{D}$. Calculate the distributional derivative of F_f.

3. (10 points) Let $\Omega = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1, y_1, y_2 > 0\}$. Find the Green's function for Ω.

4. (8 points) Consider

$$(*)\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} + u = g & x \in \partial\Omega. \end{cases}$$

(a) State the definition of a single-layer potential with moment h.

(b) In order to write the solution of (*) as a single-layer potential, what equation must h satisfy?

5. (10 points) Solve

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < \infty, t > 0 \\ u(x,0) = \phi(x) & \\ u(0,t) = g(t) & \end{cases}$$

6. (10 points) Let Ω be the triangle with vertices at $(1,0)$, $(-1,0)$ and $(0,2)$. Let λ_1 be the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Use the Comparison Principle to get an upper bound on the first eigenvalue for this eigenvalue problem. In particular, find the best upper bound on λ_1 among all rectangles contained within Ω with sides parallel to the coordinate axes.

7. (10 points) Prove Dirichlet's principle for Neumann boundary conditions. Let

$$I(w) = \frac{1}{2}\int_{\Omega} |\nabla w|^2 \, dx - \int_{\partial\Omega} gw \, dS(x).$$

Let

$$\mathcal{A} = \{w \in C^2(\Omega)\}.$$

Consider

$$(*) \begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

(a) Show that if u is a solution of (*), then

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

(b) Show that if $I(u) = \min_{w\in\mathcal{A}} I(w)$, then u is a solution of (*).

8. (12 points) Let $\Omega \equiv \{(x, y) \in \mathbb{R}^2 : 0 < x < l, 0 < y < k\}$.

(a) Find all eigenvalues and eigenfunctions for

$$\begin{cases} -\Delta \mathtt{X} = \lambda X & (x, y) \in \Omega \\ X_y(x, 0) = 0,\ X(x, k) = 0 & 0 < x < l \\ X(0, y) = 0,\ X_x(l, y) = 0 & 0 < y < k \end{cases}$$

(b) Let $X_{nm}(x, y)$ denote the eigenfunctions from part (a). Solve

$$\begin{cases} u_t - \Delta u = 0 & (x,y) \in \Omega, t > 0 \\ u(x,y,0) = \phi(x,y) & (x,y) \in \Omega \\ u_y(x,0,t) = 0,\ u(x,k,t) = 0 & 0 < x < l, t > 0 \\ u(0,y,t) = 0,\ u_x(l,y,t) = 0 & 0 < y < k, t > 0 \end{cases}$$

Express your answer in terms of $X_{nm}(x, y)$.

9. (10 points) Suppose $u \in C^2(\overline{\Omega})$ is a solution of

$$\Delta u = f \geq 0 \quad x \in \Omega.$$

Show that

$$u(x) \leq ⨍_{\partial B(x,r)} u(y)\, dy$$

for all $B(x,r) \subset \Omega$.

10. (8 points) Find the smooth function f which yields the best lower bound for $\int_0^1 (g'(x))^2 \, dx$ among functions satisfying $g(0) = 3, \ g(1) = 4$.

Scratch Paper