Math 220B Final Exam Solutions Summer 2002

1. (14 points) Let Ω be an open bounded subset of $\mathbb{R}^n$ and let V be a bounded continuous real-valued function on $\overline{\Omega}$. Consider the following Dirichlet eigenvalue problem.

$$\begin{cases} -\Delta u + V(x)u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

(a) Show that the eigenvalues are real.

Answer: Let λ be an eigenvalue. We will show that $\lambda = \overline{\lambda}$, and, therefore, the eigenvalue λ is real. First, note that if u is an eigenfunction and with eigenvalue λ of the problem above, then $\overline{u}, \overline{\lambda}$ is a solution of

$$-\Delta\overline{u} + V\overline{u} = \overline{\lambda}\overline{u}.$$

Now

$$\begin{aligned}
\lambda \int_\Omega u\overline{u}\,dx &= \int_\Omega (-\Delta u + Vu)\overline{u}\,dx \\
&= -\int_\Omega \Delta u\overline{u}\,dx + \int_\Omega Vu\overline{u}\,dx \\
&= -\int_\Omega u\Delta\overline{u}\,dx + \int_\Omega Vu\overline{u}\,dx \\
&= \int_\Omega (-V\overline{u} + \overline{\lambda}\overline{u})u\,dx + \int_\Omega Vu\overline{u}\,dx \\
&= \overline{\lambda} \int_\Omega u\overline{u}\,dx.
\end{aligned}$$

Therefore,

$$(\lambda - \overline{\lambda}) \int_\Omega |u|^2\,dx = 0,$$

which implies $\lambda = \overline{\lambda}$ or

$$\int_\Omega |u|^2\,dx = 0.$$

But, $\int_\Omega |u|^2\,dx \neq 0$, because that would imply u is the zero function, which is not an eigenfunction. Therefore, $\lambda = \overline{\lambda}$.

(b) Show that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Answer: Let X_n and X_m denote eigenfunctions corresponding to $\lambda_n \neq \lambda_m$. Therefore,

$$\begin{aligned}
\lambda_n \int_\Omega X_n X_m\,dx &= \int_\Omega (-\Delta X_n + V(x)X_n)X_m\,dx \\
&= \int_\Omega -X_n\Delta X_m\,dx + \int_\Omega V(x)X_nX_m\,dx \\
&= \int_\Omega X_n(-V(x)X_m + \lambda_m X_m)\,dx + \int_\Omega V(x)X_nX_m\,dx \\
&= \lambda_m \int_\Omega X_nX_m\,dx.
\end{aligned}$$

Therefore, we conclude that

$$(\lambda_n - \lambda_m) \int_\Omega X_n X_m \, dx = 0.$$

By assumption, $\lambda_n \neq \lambda_m$. Therefore, we conclude that $\int_\Omega X_n X_m \, dx = 0$, which means X_n and X_m are orthogonal.

(c) Show that if V is positive, then all the eigenvalues are positive.

Answer: Let λ be an eigenvalue with eigenfunction u.

$$\begin{aligned}
\lambda \int_\Omega u^2 \, dx &= \int_\Omega (-\Delta u + V(x)u)u \, dx \\
&= \int_\Omega |\nabla u|^2 \, dx + \int_\Omega V(x)u^2 \, dx \geq 0.
\end{aligned}$$

Therefore, $\lambda \geq 0$. It just remains to show that $\lambda > 0$. But, if $\lambda = 0$, then we have

$$\int_\Omega |\nabla u|^2 \, dx = 0 = \int_\Omega V(x)u^2 \, dx.$$

But, this implies that $u \equiv 0$ on Ω. However, the zero function is not an eigenfunction. Therefore, we conclude that $\lambda > 0$.

2. (8 points) Let $f(x) = H(x-1)\sin x$ where

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Define the distribution F_f associated with f such that

$$(F_f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x) \, dx$$

for all $\phi \in \mathcal{D}$. Calculate the distributional derivative of F_f.

Answer: By definition, the derivative of F_f, denoted F_f' is the distribution such that

$$(F_f', \phi) = -(F_f, \phi') \qquad \forall \phi \in \mathcal{D}.$$

Therefore,

$$\begin{aligned}
(F_f', \phi) &= -(F_f, \phi') \\
&= -\int_{-\infty}^{\infty} f(x)\phi'(x) \, dx \\
&= -\int_{-\infty}^{\infty} H(x-1)\sin(x)\phi'(x) \, dx \\
&= -\int_1^K \sin(x)\phi'(x) \, dx
\end{aligned}$$

where $\text{supp}\phi \subset \{\mathrm{x} \in \mathbb{R}, \mathrm{x} < \mathrm{K}\}$. Now integrating by parts, we have

$$\begin{aligned} -\int_1^K \sin(x)\phi'(x)\,dx &= \int_1^K \cos(x)\phi(x)\,dx - \sin(x)\phi(x)\big|_{x=1}^{x=K} \\ &= \int_1^K \cos(x)\phi(x)\,dx - [0 - \sin(1)\phi(1)] \\ &= \int_1^K \cos(x)\phi(x)\,dx + \sin(1)\phi(1). \end{aligned}$$

Therefore,

$$\boxed{(F_f', \phi) = \int_1^\infty \cos(x)\phi(x)\,dx + \sin(1)\phi(1).}$$

3. (10 points) Let $\Omega = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1, y_1, y_2 > 0\}$. Find the Green's function for Ω.

Answer: Fix $x = (x_1, x_2) \in \Omega$. Let

$$z_1 = \frac{x}{|x|^2},$$

the dual point of x. Now we need to reflect these points about the y_1 axis. Let z_2 be the reflection of x about the y_1 axis, and, let z_3 be the reflection of z_1 about the y_1 axis. That is,

$$\begin{aligned} z_2 &= (x_1, -x_2) \\ z_3 &= \frac{z_2}{|z_2|^2}. \end{aligned}$$

Now we need to reflect all four points (x, z_1, z_2, z_3) about the y_2 axis. Let

$$\begin{aligned} z_4 &= (-x_1, x_2) && \text{(reflection of } x \text{ about the } y_2 \text{ axis)} \\ z_5 &= \frac{z_4}{|z_4|^2} && \text{(reflection of } z_1 \text{ about the } y_2 \text{ axis)} \\ z_6 &= (-x_1, -x_2) && \text{(reflection of } z_2 \text{ about the } y_2 \text{ axis)} \\ z_7 &= \frac{z_6}{|z_6|^2} && \text{(reflection of } z_3 \text{ about the } y_2 \text{ axis)} \end{aligned}$$

Then

$$\boxed{\begin{aligned} G(x, y) = \Phi(y - x) - \Phi(|x|(y - z_1)) - \Phi(y - z_2) + \Phi(|x|(y - z_3)) - \Phi(y - z_4) \\ + \Phi(|x|(y - z_5)) + \Phi(y - z_6) - \Phi(|x|(y - z_7)). \end{aligned}}$$

4. (8 points) Consider

$$(*) \begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} + u = g & x \in \partial\Omega. \end{cases}$$

(a) State the definition of a single-layer potential with moment h.

Answer:

$$\boxed{\overline{u}(x) = -\int_{\Omega} h(y)\Phi(y-x)\,dS(y).}$$

(b) In order to write the solution of (*) as a single-layer potential, what equation must h satisfy?

Answer: Fix $x_0 \in \partial\Omega$. Let $\nu(x_0)$ be the outer unit normal to $\partial\Omega$ at x_0. For all $t < 0$ such that $x_0 + t\nu(x_0) \in \Omega$, let

$$i^{x_0}(t) = \nabla\overline{u}(x_0 + t\nu(x_0)) \cdot \nu(x_0).$$

We will say the boundary condition is satisfied if

$$\lim_{t\to 0^-} i^{x_0}(t) + \overline{u}(x) = g(x_0).$$

for all $x \in \Omega$, $x_0 \in \partial\Omega$. Now

$$\lim_{t\to 0^-} i^{x_0}(t) = -\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_x}(x_0 - y)\,dS(y).$$

In addition, we recall that a single-layer potential is continuous for all $x \in \mathbb{R}^n$. Therefore, in order to find a solution as a single-layer potential, we need h to satisfy

$$\boxed{-\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_x}(x_0 - y)\,dS(y) - \int_{\partial\Omega} h(y)\Phi(y - x_0)\,dS(y) = g(x_0)}$$

for all $x_0 \in \partial\Omega$.

5. (10 points) Solve

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < \infty, t > 0 \\ u(x,0) = \phi(x) & \\ u(0,t) = g(t) & \end{cases}$$

Answer: Suppose u is the solution. Let $v = u - g(t)$ for $x > 0$. Then v is a solution of

$$\begin{cases} v_t - v_{xx} = -g'(t) & 0 < x < \infty \\ v(x,0) = \phi(x) - g(0) & 0 < x < \infty. \end{cases}$$

Now extend v to the negative axis, by introducing a new function $\widetilde{v}$ such that

$$\widetilde{v}(x) = \begin{cases} v(x) & x > 0 \\ -v(-x) & x < 0. \end{cases}$$

Therefore, $\widetilde{v}$ is a solution of

$$\begin{cases} \widetilde{v} - \widetilde{v}_{xx} = f(x,t) & -\infty < x < \infty \\ \widetilde{v}(x,0) = h(x) & \end{cases}$$

where

$$f(x,t) = \begin{cases} -g'(t) & x > 0 \\ g'(t) & x < 0 \end{cases}$$

and

$$h(x) = \begin{cases} \phi(x) - g(0) & x > 0 \\ -\phi(-x) + g(0) & x < 0 \end{cases}$$

Now by Duhamel's principle, the solution of the IVP for $\widetilde{v}$ is given by

$$\widetilde{v}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}} h(y)\,dy + \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4(t-s)}} f(y,s)\,dy\,ds$$

Now v is the restriction of $\widetilde{v}$ to the positive x-axis. Then $v = u - g$ implies $u = v + g$. Therefore,

$$\boxed{u(x,t) = g(t) + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}} h(y)\,dy + \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4(t-s)}} f(y,s)\,dy\,ds}$$

for f and h defined above.

6. (10 points) Let Ω be the triangle with vertices at $(1,0)$, $(-1,0)$ and $(0,2)$. Let λ_1 be the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Use the Comparison Principle to get an upper bound on the first eigenvalue for this eigenvalue problem. In particular, find the best upper bound on λ_1 among all rectangles contained within Ω with sides parallel to the coordinate axes.

Answer: For any rectangle R contained within Ω with sides parallel to the axes, and vertices $(x,0), (-x,0), (x,y), (-x,y)$, the eigenvalues of R are given by

$$\lambda_{nm}(R) = \left(\frac{n\pi}{2x}\right)^2 + \left(\frac{m\pi}{y}\right)^2.$$

For any rectangle contained within Ω, we can get a better estimate if we extend the rectangle so that its vertices intersect the boundary. Therefore, we want to minimize the first eigenvalue for all rectangles contained within Ω such that $y = 2 - 2x$. That is, we want to minimize the function

$$f(x) = \frac{1}{(2x)^2} + \frac{1}{y(x)^2} = \frac{1}{4x^2} + \frac{1}{(2-2x)^2} = \frac{1}{4}\left[\frac{1}{x^2} + \frac{1}{(1-x)^2}\right].$$

For simplicity, we neglect the coefficient $1/4$. We just need to minimize

$$g(x) = \frac{1}{x^2} + \frac{1}{(1-x)^2}.$$

We look for critical points.

$$g'(x) = \frac{-2}{x^3} + \frac{2}{(1-x)^3}.$$

Now $g'(x) = 0$ implies

$$\frac{1}{(1-x)^3} = \frac{1}{x^3}$$

or

$$x^3 = (1-x)^3 \implies x = 1-x \implies x = \frac{1}{2}.$$

We see that $x = 1/2$ minimizes g. Therefore, the best upper bound on the first eigenvalue of Ω given by rectangles with sides parallel to the coordinate axes is the rectangle with vertices $(1/2, 0)$, $(-1/2, 0)$, $(1/2, 1)$, $(-1/2, 1)$. Denote this rectangle by R^*. Therefore,

$$\boxed{\lambda_1(R^*) = 2\pi^2 \geq \lambda_1(\Omega).}$$

7. (10 points) Prove Dirichlet's principle for Neumann boundary conditions. Let

$$I(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_{\partial\Omega} gw\,dS(x).$$

Let

$$\mathcal{A} = \{w \in C^2(\Omega)\}.$$

Consider

$$(*)\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

(a) Show that if u is a solution of (*), then

$$I(u) = \min_{w\in\mathcal{A}} I(w).$$

Answer: Suppose u is a solution of (*). Let $w \in \mathcal{A}$.

$$\begin{aligned}
0 &= \int_\Omega \Delta u(u-w)\,dx \\
&= -\int_\Omega \nabla u \cdot \nabla(u-w)\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(u-w)\,dS(x) \\
&= -\int_\Omega |\nabla u|^2\,dx + \int_\Omega \nabla u \cdot \nabla w\,dx + \int_{\partial\Omega} gu\,dS(x) - \int_{\partial\Omega} gw\,dS(x) \\
&\leq -\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \int_{\partial\Omega} gu\,dS(x) \\
&\quad - \int_{\partial\Omega} gw\,dS(x).
\end{aligned}$$

Therefore, we conclude that

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_{\partial\Omega} gu\,dS(x) \le \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_{\partial\Omega} gw\,dS(x).$$

But, w is arbitrary in $\mathcal{A}$. Therefore, we conclude that

$$I(u) = \min_{w\in\mathcal{A}}.$$

(b) Show that if $I(u) = \min_{w\in\mathcal{A}} I(w)$, then u is a solution of (*).

Answer: Let w be an arbitrary function in $\mathcal{A}$. Let

$$i(\epsilon) = I(u + \epsilon v).$$

If u is a minimizer of I, then i must have a local minimum at $\epsilon = 0$. Therefore, $i'(0) = 0$. Now

$$\begin{aligned} i'(\epsilon) &= \frac{d}{d\epsilon} I(u+\epsilon v) \\ &= \frac{d}{d\epsilon}\left(\frac{1}{2}\int_\Omega |\nabla u + \epsilon\nabla v|^2\,dx - \int_{\partial\Omega} g(u+\epsilon v)\,dS(x)\right) \\ &= \int_\Omega \nabla u\cdot\nabla v + \epsilon|\nabla v|^2\,dx - \int_{\partial\Omega} gv\,dS(x). \end{aligned}$$

Therefore, $i'(0) = 0$ implies

$$i'(0) = \int_\Omega \nabla u\cdot\nabla v\,dx - \int_{\partial\Omega} gv\,dS(x) = 0.$$

This implies

$$-\int_\Omega \Delta uv\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial\nu} v\,dS(x) - \int_{\partial\Omega} gv\,dS(x) = 0$$

or

$$(**)\int_\Omega \Delta uv\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial\nu} v\,dS(x) - \int_{\partial\Omega} gv\,dS(x).$$

Now this is true for all $v \in \mathcal{A}$. Let $\widetilde{\mathcal{A}}$ be the subset of $\mathcal{A}$ such that

$$\widetilde{\mathcal{A}} = \{w \in C^2(\Omega) : w = 0 \text{ for } x \in \partial\Omega\}.$$

Now (**) is true for $v \in \widetilde{\mathcal{A}}$ as well. But, for $v \in \widetilde{\mathcal{A}}$ the right-hand side of (**) vanishes. Therefore, we conclude that for all $v \in \widetilde{\mathcal{A}}$,

$$\int_\Omega \Delta uv\,dx = 0.$$

But, this is enough to conclude that $\Delta u = 0$. We just need to show that $\partial u/\partial\nu = g$ for $x \in \partial\Omega$. Now the left-hand side of (**) vanishes. Therefore, for all $v \in \mathcal{A}$, we have

$$\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu} - g\right) v\,dS(x) = 0.$$

but, since this is true for all $v \in \mathcal{A}$, we can conclude that $\partial u/\partial\nu = g$ for $x \in \partial\Omega$. Therefore, u is a solution of the Neumann problem (*).

8. (12 points) Let $\Omega \equiv \{(x,y) \in \mathbb{R}^2 : 0 < x < l, 0 < y < k\}$.

(a) Find all eigenvalues and eigenfunctions for

$$\begin{cases} -\Delta X = \lambda X & (x,y) \in \Omega \\ X_y(x,0) = 0,\ X(x,k) = 0 & 0 < x < l \\ X(0,y) = 0,\ X_x(l,y) = 0 & 0 < y < k \end{cases}$$

Answer: Using separation of variables, we look for eigenfunctions of the form $X(x)Y(y)$. Plugging this into the eigenvalue problem, we have

$$-X''Y - XY'' = \lambda XY.$$

Dividing by XY, we have

$$-\frac{X''}{X} - \frac{Y''}{Y} = \lambda.$$

This implies

$$-\frac{X''}{X} = \frac{Y''}{Y} + \lambda = \mu.$$

Therefore, we need to solve the eigenvalue problem

$$\begin{cases} -X'' = \mu X & 0 < x < l \\ X(0) = 0, X'(l) = 0. \end{cases}$$

If $\mu = \beta^2 > 0$, we have

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

The boundary condition

$$X(0) = 0 \implies A = 0.$$

The boundary condition

$$X'(l) = 0 \implies \cos(\beta l) = 0 \implies \beta l = \left(n + \frac{1}{2}\right)\pi.$$

Therefore, $\mu_n = \beta_n^2 = \left(\left(n + \frac{1}{2}\right)\frac{\pi}{l}\right)^2$ and $X_n(x) = \sin(\beta_n x)$.
Then, we need to solve our equation for Y. In particular, we need to solve

$$\frac{Y''}{Y} + \lambda = \mu \implies \frac{Y''}{Y} = \mu - \lambda = -\gamma.$$

This leads us to the eigenvalue problem

$$\begin{cases} -Y'' = \gamma Y & 0 < y < k \\ Y'(0) = 0 = Y(k). \end{cases}$$

If $\gamma = \alpha^2 > 0$, then

$$Y(y) = A\cos(\alpha y) + B\sin(\alpha y).$$

The boundary condition

$$Y'(0) = 0 \implies B = 0.$$

The boundary condition

$$Y(k) = 0 \implies \cos(\alpha k) = 0 \implies \alpha k = \left(m + \frac{1}{2}\right)\pi.$$

Therefore, $\gamma_m = \alpha_m^2 = \left(\left(m + \frac{1}{2}\right)\frac{\pi}{k}\right)^2$ and $Y_m(y) = \cos(\alpha_m y)$.
Therefore, to conclude our eigenfunctions and corresponding eigenvalues are given by

$$\boxed{\begin{array}{l} X_{nm}(x,y) = \sin(\beta_n x)\cos(\alpha_m y) \\ \quad \text{where } \beta_n = \left(n + \frac{1}{2}\right)\frac{\pi}{l} \text{ and } \alpha_m = \left(m + \frac{1}{2}\right)\frac{\pi}{k} \\ \lambda_{nm} = \mu_n + \gamma_n = \beta_n^2 + \alpha_m^2 \end{array}}$$

(b) Let $X_{nm}(x,y)$ denote the eigenfunctions from part (a). Solve

$$\begin{cases} u_t - \Delta u = 0 & (x,y) \in \Omega, t > 0 \\ u(x,y,0) = \phi(x,y) & (x,y) \in \Omega \\ u_y(x,0,t) = 0,\ u(x,k,t) = 0 & 0 < x < l, t > 0 \\ u(0,y,t) = 0,\ u_x(l,y,t) = 0 & 0 < y < k, t > 0 \end{cases}$$

Express your answer in terms of $X_{nm}(x,y)$.

Answer: We look for a solution of the form $T(t)X(x,y)$. Plugging this into the PDE, we have

$$T'X - T\Delta X = 0.$$

Dividing by TX, we have

$$\frac{T'}{T} - \frac{\Delta X}{X} = 0,$$

which implies

$$\frac{T'}{T} = \frac{\Delta X}{X} = -\lambda.$$

Let $X_{nm}(x,y), \lambda_{nm}$ denote the eigenfunctions and corresponding eigenvalues of

$$\begin{cases} -\Delta X = \lambda X & (x,y) \in \Omega \\ X(0,y) = 0 = X_x(l,y) & 0 < y < k \\ X_y(x,0) = 0 = X(x,k) & 0 < x < l. \end{cases}$$

Our solution for our equation for T_{nm} is

$$T_{nm}(t) = C_{nm}e^{-\lambda_{nm}t}.$$

Therefore, our solution is given by

$$\boxed{u(x,y,t) = \sum_{n,m=0}^{\infty} C_{nm}X_{nm}(x,y)e^{-\lambda_{nm}t}}$$

where

$$\boxed{C_{nm} = \frac{\langle \phi, X_{nm} \rangle}{\langle X_{nm}, X_{nm} \rangle} = \frac{\int_0^k \int_0^l \phi X_{nm}\, dx\, dy}{\int_0^k \int_0^l X_{nm}^2\, dx\, dy}.}$$

9. (10 points) Suppose $u \in C^2(\overline{\Omega})$ is a solution of

$$\Delta u = f \geq 0 \quad x \in \Omega.$$

Show that

$$u(x) \leq \fint_{\partial B(x,r)} u(y)\, dy$$

for all $B(x,r) \subset \Omega$.

Answer: Define the function

$$\phi(r) = \fint_{\partial B(x,r)} u(y)\, dS(y)$$

for $r > 0$ and $\phi(0) = u(x)$. Using the assumption that u is continuous, we conclude that ϕ is continuous. We now look at $\phi'(r)$.

$$\begin{aligned}
\phi'(r) &= \frac{d}{dr} \fint_{\partial B(x,r)} u(y)\, dS(y) \\
&= \frac{d}{dr} \fint_{\partial B(0,1)} u(x + rz)\, dS(z) \\
&= \fint_{\partial B(0,1)} \nabla u(x + rz) \cdot z\, dS(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r}\, dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, dS(y) \\
&= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, dS(y) \\
&= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u\, dy \\
&\geq 0.
\end{aligned}$$

Therefore, ϕ' is an increasing function of r. Therefore, we conclude that $\phi(0) \leq \phi(r)$. Therefore, we have

$$u(x) \leq \fint_{\partial B(x,r)} u(y)\, dS(y),$$

as claimed.

10. (8 points) Find the smooth function f which yields the best lower bound for $\int_0^1 (g'(x))^2\,dx$ among functions satisfying $g(0) = 3$, $g(1) = 4$.

Answer: By Dirichlet's principle, we know that the function which minimizes $I(w) = \int_\Omega |\nabla w|^2\,dx$ subject to certain boundary conditions is the harmonic function on Ω which satisfies those boundary conditions. Therefore, we look for the solution to

$$\begin{cases} g'' = 0 & 0 < x < 1 \\ g(0) = 3, g(1) = 4. \end{cases}$$

Clearly, the harmonic functions on an interval are linear functions. That is,

$$g(x) = A + Bx.$$

The boundary condition

$$g(0) = 3 \implies A = 3.$$

The boundary condition

$$g(1) = 4 \implies B = 1.$$

Therefore, the solution is

$$\boxed{g(x) = 3 + x.}$$