1. (14 points) Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let $V$ be a bounded continuous real-valued function on $\overline{\Omega}$. Consider the following Dirichlet eigenvalue problem.

\[
\begin{cases}
-\Delta u + V(x)u = \lambda u & x \in \Omega \\
u = 0 & x \in \partial \Omega
\end{cases}
\]

(a) Show that the eigenvalues are real.

**Answer:** Let $\lambda$ be an eigenvalue. We will show that $\lambda = \overline{\lambda}$, and, therefore, the eigenvalue $\lambda$ is real. First, note that if $u$ is an eigenfunction and with eigenvalue $\lambda$ of the problem above, then $\overline{u}, \overline{\lambda}$ is a solution of

\[-\Delta \overline{u} + V\overline{u} = \overline{\lambda} \overline{u}.
\]

Now

\[
\lambda \int_{\Omega} u \overline{u} \, dx = \int_{\Omega} (-\Delta u + V u) \overline{u} \, dx
\]

\[
= -\int_{\Omega} \Delta u \overline{u} \, dx + \int_{\Omega} V u \overline{u} \, dx
\]

\[
= -\int_{\Omega} u \Delta \overline{u} \, dx + \int_{\Omega} V u \overline{u} \, dx
\]

\[
= \int_{\Omega} (-V \overline{u} + \overline{\lambda} \overline{u}) u \, dx + \int_{\Omega} V u \overline{u} \, dx
\]

\[
= \lambda \int_{\Omega} u \overline{u} \, dx.
\]

Therefore,

\[
(\lambda - \overline{\lambda}) \int_{\Omega} |u|^2 \, dx = 0,
\]

which implies $\lambda = \overline{\lambda}$ or

\[
\int_{\Omega} |u|^2 \, dx = 0.
\]

But, $\int_{\Omega} |u|^2 \, dx \neq 0$, because that would imply $u$ is the zero function, which is not an eigenfunction. Therefore, $\lambda = \overline{\lambda}$.

(b) Show that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

**Answer:** Let $X_n$ and $X_m$ denote eigenfunctions corresponding to $\lambda_n \neq \lambda_m$. Therefore,

\[
\lambda_n \int_{\Omega} X_n X_m \, dx = \int_{\Omega} (-\Delta X_n + V(x)X_n)X_m \, dx
\]

\[
= \int_{\Omega} -X_n \Delta X_m \, dx + \int_{\Omega} V(x)X_n X_m \, dx
\]

\[
= \int_{\Omega} X_n (-V(x)X_m + \lambda_m X_m) \, dx + \int_{\Omega} V(x)X_n X_m \, dx
\]

\[
= \lambda_m \int_{\Omega} X_n X_m \, dx.
\]
Therefore, we conclude that

\[(\lambda_n - \lambda_m) \int_{\Omega} X_n X_m \, dx = 0.\]

By assumption, \(\lambda_n \neq \lambda_m\). Therefore, we conclude that \(\int_{\Omega} X_n X_m \, dx = 0\), which means \(X_n\) and \(X_m\) are orthogonal.

(c) Show that if \(V\) is positive, then all the eigenvalues are positive.

**Answer:** Let \(\lambda\) be an eigenvalue with eigenfunction \(u\).

\[
\lambda \int_{\Omega} u^2 \, dx = \int_{\Omega} (-\Delta u + V(x)u)u \, dx
= \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} V(x)u^2 \, dx \geq 0.
\]

Therefore, \(\lambda \geq 0\). It just remains to show that \(\lambda > 0\). But, if \(\lambda = 0\), then we have

\[
\int_{\Omega} |\nabla u|^2 \, dx = 0 = \int_{\Omega} V(x)u^2 \, dx.
\]

But, this implies that \(u \equiv 0\) on \(\Omega\). However, the zero function is not an eigenfunction. Therefore, we conclude that \(\lambda > 0\).

2. (8 points) Let \(f(x) = H(x - 1)\sin x\) where

\[
H(x) = \begin{cases}
1 & x \geq 0 \\
0 & x < 0.
\end{cases}
\]

Define the distribution \(F_f\) associated with \(f\) such that

\[
(F_f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x) \, dx
\]

for all \(\phi \in \mathcal{D}\). Calculate the distributional derivative of \(F_f\).

**Answer:** By definition, the derivative of \(F_f\), denoted \(F_f'\) is the distribution such that

\[
(F_f, \phi) = -(F_f', \phi') \quad \forall \phi \in \mathcal{D}.
\]

Therefore,

\[
(F_f', \phi) = -(F_f, \phi')
= -\int_{-\infty}^{\infty} f(x)\phi'(x) \, dx
= -\int_{-\infty}^{\infty} H(x - 1)\sin(x)\phi'(x) \, dx
= -\int_{1}^{\infty} \sin(x)\phi'(x) \, dx
\]
where suppφ ⊂ \{x ∈ ℝ, x < K\}. Now integrating by parts, we have
\[-\int_1^K \sin(x)\phi'(x) dx = \int_1^K \cos(x)\phi(x) dx - \sin(x)\phi(x)\bigg|_{x=1}^{x=K}\]
\[= \int_1^K \cos(x)\phi(x) dx - [0 - \sin(1)\phi(1)]\]
\[= \int_1^K \cos(x)\phi(x) dx + \sin(1)\phi(1).\]

Therefore,
\[(F_f', \phi) = \int_1^\infty \cos(x)\phi(x) dx + \sin(1)\phi(1).\]

3. (10 points) Let Ω = \{(y_1, y_2) ∈ ℝ^2 : y_1^2 + y_2^2 < 1, y_1, y_2 > 0\}. Find the Green’s function for Ω.

**Answer:** Fix x = (x_1, x_2) ∈ Ω. Let
\[z_1 = \frac{x}{|x|^2},\]
the dual point of x. Now we need to reflect these points about the y_1 axis. Let z_2 be the reflection of x about the y_1 axis, and, let z_3 be the reflection of z_1 about the y_1 axis. That is,
\[z_2 = (x_1, -x_2),\]
\[z_3 = \frac{z_2}{|z_2|^2}.\]

Now we need to reflect all four points \((x, z_1, z_2, z_3)\) about the y_2 axis. Let
\[z_4 = (-x_1, x_2)\]  (reflection of x about the y_2 axis)
\[z_5 = \frac{z_4}{|z_4|^2}\]  (reflection of z_1 about the y_2 axis)
\[z_6 = (-x_1, -x_2)\]  (reflection of z_2 about the y_2 axis)
\[z_7 = \frac{z_6}{|z_6|^2}\]  (reflection of z_3 about the y_2 axis)

Then
\[G(x, y) = \Phi(y - x) - \Phi(|x|(y - z_1)) - \Phi(y - z_2) + \Phi(|x|(y - z_3)) - \Phi(y - z_4)\]
\[+ \Phi(|x|(y - z_5)) + \Phi(y - z_6) - \Phi(|x|(y - z_7)).\]

4. (8 points) Consider
\[\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} + u = g & x \in \partial \Omega. \end{cases}\]
(a) State the definition of a single-layer potential with moment \( h \).

**Answer:**

\[
\overline{u}(x) = -\int_{\Omega} h(y)\Phi(y-x)\,dS(y).
\]

(b) In order to write the solution of (\( \ast \)) as a single-layer potential, what equation must \( h \) satisfy?

**Answer:** Fix \( x_0 \in \partial \Omega \). Let \( \nu(x_0) \) be the outer unit normal to \( \partial \Omega \) at \( x_0 \). For all \( t < 0 \) such that \( x_0 + t\nu(x_0) \in \Omega \), let

\[
i^{x_0}(t) = \nabla \overline{u}(x_0 + t\nu(x_0)) \cdot \nu(x_0).
\]

We will say the boundary condition is satisfied if

\[
\lim_{t \to 0^-} i^{x_0}(t) + \overline{u}(x) = g(x_0).
\]

for all \( x \in \Omega \), \( x_0 \in \partial \Omega \). Now

\[
\lim_{t \to 0^-} i^{x_0}(t) = -\frac{1}{2} h(x_0) - \int_{\partial \Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0 - y)\,dS(y).
\]

In addition, we recall that a single-layer potential is continuous for all \( x \in \mathbb{R}^n \). Therefore, in order to find a solution as a single-layer potential, we need \( h \) to satisfy

\[
-\frac{1}{2} h(x_0) - \int_{\partial \Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0 - y)\,dS(y) - \int_{\partial \Omega} h(y)\Phi(y - x_0)\,dS(y) = g(x_0)
\]

for all \( x_0 \in \partial \Omega \).

5. (10 points) Solve

\[
\begin{aligned}
&u_t - u_{xx} = 0 & 0 < x < \infty, t > 0 \\
u(x, 0) &= \phi(x) \\
u(0, t) &= g(t)
\end{aligned}
\]

**Answer:** Suppose \( u \) is the solution. Let \( v = u - g(t) \) for \( x > 0 \). Then \( v \) is a solution of

\[
\begin{aligned}
v_t - v_{xx} &= -g'(t) & 0 < x < \infty \\
v(x, 0) &= \phi(x) - g(0) & 0 < x < \infty.
\end{aligned}
\]

Now extend \( v \) to the negative axis, by introducing a new function \( \tilde{v} \) such that

\[
\tilde{v}(x) = \begin{cases} 
 v(x) & x > 0 \\
 -v(-x) & x < 0.
\end{cases}
\]

Therefore, \( \tilde{v} \) is a solution of

\[
\begin{aligned}
\tilde{v} - \tilde{v}_{xx} &= f(x, t) & -\infty < x < \infty \\
\tilde{v}(x, 0) &= \tilde{h}(x)
\end{aligned}
\]
where
\[ f(x, t) = \begin{cases} -g'(t) & x > 0 \\ g'(t) & x < 0 \end{cases} \]
and
\[ h(x) = \begin{cases} \phi(x) - g(0) & x > 0 \\ -\phi(-x) + g(0) & x < 0 \end{cases} \]

Now by Duhamel’s principle, the solution of the IVP for \( \tilde{v} \) is given by
\[
\tilde{v}(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} h(y) \, dy + \int_{0}^{t} \frac{1}{\sqrt{4\pi (t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}} f(y, s) \, dy \, ds
\]

Now \( v \) is the restriction of \( \tilde{v} \) to the positive \( x \)-axis. Then \( v = u - g \) implies \( u = v + g \). Therefore,
\[
\begin{aligned}
u(x, t) &= g(t) + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} h(y) \, dy + \int_{0}^{t} \frac{1}{\sqrt{4\pi (t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}} f(y, s) \, dy \, ds
\end{aligned}
\]
for \( f \) and \( h \) defined above.

6. (10 points) Let \( \Omega \) be the triangle with vertices at \((1, 0), (-1, 0)\) and \((0, 2)\). Let \( \lambda_1 \) be the first eigenvalue of
\[
\begin{aligned}
-\Delta u = \lambda u \\
u = 0
\end{aligned} \quad x \in \Omega \\
x \in \partial \Omega.
\]
Use the Comparison Principle to get an upper bound on the first eigenvalue for this eigenvalue problem. In particular, find the best upper bound on \( \lambda_1 \) among all rectangles contained within \( \Omega \) with sides parallel to the coordinate axes.

**Answer:** For any rectangle \( R \) contained within \( \Omega \) with sides parallel to the axes, and vertices \((x, 0), (-x, 0), (x, y), (-x, y)\), the eigenvalues of \( R \) are given by
\[
\lambda_{nm}(R) = \left( \frac{n\pi}{2x} \right)^2 + \left( \frac{m\pi}{y} \right)^2.
\]

For any rectangle contained within \( \Omega \), we can get a better estimate if we extend the rectangle so that its vertices intersect the boundary. Therefore, we want to minimize the first eigenvalue for all rectangles contained within \( \Omega \) such that \( y = 2 - 2x \). That is, we want to minimize the function
\[
f(x) = \frac{1}{(2x)^2} + \frac{1}{y(x)^2} = \frac{1}{4x^2} + \frac{1}{(2-2x)^2} = \frac{1}{4} \left[ \frac{1}{x^2} + \frac{1}{(1-x)^2} \right].
\]

For simplicity, we neglect the coefficient 1/4. We just need to minimize
\[
g(x) = \frac{1}{x^2} + \frac{1}{(1-x)^2}.
\]
We look for critical points.

\[ g'(x) = \frac{-2}{x^3} + \frac{2}{(1-x)^3}. \]

Now \( g'(x) = 0 \) implies

\[ \frac{1}{(1-x)^3} = \frac{1}{x^3} \]

or

\[ x^3 = (1-x)^3 \implies x = 1 - x \implies x = \frac{1}{2}. \]

We see that \( x = 1/2 \) minimizes \( g \). Therefore, the best upper bound on the first eigenvalue of \( \Omega \) given by rectangles with sides parallel to the coordinate axes is the rectangle with vertices \((1/2, 0), (-1/2, 0), (1/2, 1), (-1/2, 1)\). Denote this rectangle by \( R^* \). Therefore,

\[ \lambda_1(R^*) = 2\pi^2 \geq \lambda_1(\Omega). \]

7. (10 points) Prove Dirichlet’s principle for Neumann boundary conditions. Let

\[ I(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx - \int_{\partial\Omega} gw \, dS(x). \]

Let

\[ \mathcal{A} = \{ w \in C^2(\Omega) \}. \]

Consider

\[ (*) \begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial \Omega \end{cases} \]

(a) Show that if \( u \) is a solution of \( (*) \), then

\[ I(u) = \min_{w \in \mathcal{A}} I(w). \]

**Answer:** Suppose \( u \) is a solution of \( (*) \). Let \( w \in \mathcal{A} \).

\[ 0 = \int_\Omega \Delta u (u-w) \, dx \]

\[ = -\int_\Omega \nabla u \cdot \nabla (u-w) \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} (u-w) \, dS(x) \]

\[ = -\int_\Omega |\nabla u|^2 \, dx + \int_\Omega \nabla u \cdot \nabla w \, dx + \int_{\partial\Omega} gu \, dS(x) - \int_{\partial\Omega} gw \, dS(x) \]

\[ \leq -\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx + \int_{\partial\Omega} gu \, dS(x) \]

\[ - \int_{\partial\Omega} gw \, dS(x). \]
Therefore, we conclude that
\[
\frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\partial \Omega} g u \, dS(x) \leq \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx - \int_{\partial \Omega} g w \, dS(x).
\]
But, \( w \) is arbitrary in \( \mathcal{A} \). Therefore, we conclude that
\[
I(u) = \min_{w \in \mathcal{A}} I(w).
\]

(b) Show that if \( I(u) = \min_{w \in \mathcal{A}} I(w) \), then \( u \) is a solution of (*).

\textbf{Answer:} Let \( w \) be an arbitrary function in \( \mathcal{A} \). Let
\[
i(\epsilon) = I(u + \epsilon v).
\]
If \( u \) is a minimizer of \( I \), then \( i \) must have a local minimum at \( \epsilon = 0 \). Therefore, \( i'(0) = 0 \). Now
\[
i'(\epsilon) = \frac{d}{d\epsilon} I(u + \epsilon v)
= \frac{d}{d\epsilon} \left( \frac{1}{2} \int_{\Omega} |\nabla u + \epsilon \nabla v|^2 \, dx - \int_{\partial \Omega} g(u + \epsilon v) \, dS(x) \right)
= \int_{\Omega} \nabla u \cdot \nabla v + \epsilon |\nabla v|^2 \, dx - \int_{\partial \Omega} g v \, dS(x).
\]
Therefore, \( i'(0) = 0 \) implies
\[
i'(0) = \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial \Omega} g v \, dS(x) = 0.
\]
This implies
\[
-\int_{\Omega} \Delta uv \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} v \, dS(x) - \int_{\partial \Omega} g v \, dS(x) = 0
\]
or
\[
(**) \int_{\Omega} \Delta uv \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} v \, dS(x) - \int_{\partial \Omega} g v \, dS(x).
\]
Now this is true for all \( v \in \mathcal{A} \). Let \( \tilde{\mathcal{A}} \) be the subset of \( \mathcal{A} \) such that
\[
\tilde{\mathcal{A}} = \{ w \in C^2(\Omega) : w = 0 \text{ for } x \in \partial \Omega \}.
\]
Now (**) is true for \( v \in \tilde{\mathcal{A}} \) as well. But, for \( v \in \tilde{\mathcal{A}} \) the right-hand side of (**) vanishes. Therefore, we conclude that for all \( v \in \tilde{\mathcal{A}} \),
\[
\int_{\Omega} \Delta uv \, dx = 0.
\]
But, this is enough to conclude that \( \Delta u = 0 \). We just need to show that \( \frac{\partial u}{\partial \nu} = g \) for \( x \in \partial \Omega \). Now the left-hand side of (***) vanishes. Therefore, for all \( v \in \mathcal{A} \), we have
\[
\int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} - g \right) v \, dS(x) = 0.
\]
but, since this is true for all \( v \in \mathcal{A} \), we can conclude that \( \frac{\partial u}{\partial \nu} = g \) for \( x \in \partial \Omega \). Therefore, \( u \) is a solution of the Neumann problem (*).
8. (12 points) Let \( \Omega \equiv \{(x, y) \in \mathbb{R}^2 : 0 < x < l, 0 < y < k\} \).

(a) Find all eigenvalues and eigenfunctions for
\[
\begin{aligned}
-\Delta X &= \lambda X & (x, y) &\in \Omega \\
X_y(x, 0) = 0, X(x, k) &= 0 & 0 < x < l \\
X(0, y) = 0, X_x(l, y) &= 0 & 0 < y < k
\end{aligned}
\]

\textbf{Answer:} Using separation of variables, we look for eigenfunctions of the form \( X(x)Y(y) \). Plugging this into the eigenvalue problem, we have
\[-X''Y - XY'' = \lambda XY.\]

Dividing by \( XY \), we have
\[-\frac{X''}{X} - \frac{Y''}{Y} = \lambda.\]

This implies
\[-\frac{X''}{X} = \frac{Y''}{Y} + \lambda = \mu.\]

Therefore, we need to solve the eigenvalue problem
\[
\begin{aligned}
-\frac{X''}{X} &= \mu X & 0 < x < l \\
X(0) &= 0, X'(l) = 0.
\end{aligned}
\]

If \( \mu = \beta^2 > 0 \), we have
\[X(x) = A \cos(\beta x) + B \sin(\beta x).\]

The boundary condition
\[X(0) = 0 \implies A = 0.\]

The boundary condition
\[X'(l) = 0 \implies \cos(\beta l) = 0 \implies \beta l = \left( n + \frac{1}{2} \right) \pi.\]

Therefore, \( \mu_n = \beta^2_n = \left( (n + \frac{1}{2}) \frac{\pi}{l} \right)^2 \) and \( X_n(x) = \sin(\beta_n x) \).

Then, we need to solve our equation for \( Y \). In particular, we need to solve
\[\frac{Y''}{Y} + \lambda = \mu \implies \frac{Y''}{Y} = \mu - \lambda = -\gamma.\]

This leads us to the eigenvalue problem
\[
\begin{aligned}
-Y'' &= \gamma Y & 0 < y < k \\
Y'(0) &= 0 = Y(k).
\end{aligned}
\]

If \( \gamma = \alpha^2 > 0 \), then
\[Y(y) = A \cos(\alpha y) + B \sin(\alpha y).\]
The boundary condition
\[ Y'(0) = 0 \implies B = 0. \]

The boundary condition
\[ Y(k) = 0 \implies \cos(\alpha k) = 0 \implies \alpha k = \left( m + \frac{1}{2} \right) \pi. \]

Therefore, \( \gamma_m = \alpha_m^2 = \left( m + \frac{1}{2} \right) \frac{\pi}{k} \) and \( Y_m(y) = \cos(\alpha_m y). \)

Therefore, to conclude our eigenfunctions and corresponding eigenvalues are given by
\[
\begin{align*}
X_{nm}(x, y) &= \sin(\beta_n x) \cos(\alpha_m y) \\
\text{where } \beta_n &= \left( n + \frac{1}{2} \right) \pi \quad \text{and } \alpha_m = \left( m + \frac{1}{2} \right) \frac{\pi}{k} \\
\lambda_{nm} &= \mu_n + \gamma_n = \beta_n^2 + \alpha_m^2
\end{align*}
\]

(b) Let \( X_{nm}(x, y) \) denote the eigenfunctions from part (a). Solve
\[
\begin{cases}
\begin{align*}
u_t - \Delta u &= 0 \quad &\text{if } (x, y) \in \Omega, t > 0 \\
u(x, y, 0) &= \phi(x, y) \quad &\text{if } (x, y) \in \Omega \\
u_y(x, 0, t) &= 0, \quad u(x, k, t) = 0 \quad &0 < x < l, t > 0 \\
u(0, y, t) &= 0, \quad u(l, y, t) = 0 \quad &0 < y < k, t > 0
\end{align*}
\end{cases}
\]

Express your answer in terms of \( X_{nm}(x, y) \).

**Answer:** We look for a solution of the form \( T(t)X(x, y) \). Plugging this into the PDE, we have
\[ T'X - T\Delta X = 0. \]

Dividing by \( TX \), we have
\[ \frac{T'}{T} - \frac{\Delta X}{X} = 0, \]

which implies
\[ \frac{T'}{T} = -\lambda. \]

Let \( X_{nm}(x, y), \lambda_{nm} \) denote the eigenfunctions and corresponding eigenvalues of
\[
\begin{cases}
\begin{align*}
-\Delta X &= \lambda X \quad &\text{if } (x, y) \in \Omega \\
X(0, y) &= 0 = X_x(l, y) \quad &0 < y < k \\
X_y(x, 0) &= 0 = X(x, k) \quad &0 < x < l.
\end{align*}
\end{cases}
\]

Our solution for our equation for \( T_{nm} \) is
\[ T_{nm}(t) = C_{nm} e^{-\lambda_{nm} t}. \]

Therefore, our solution is given by
\[
\begin{align*}
u(x, y, t) &= \sum_{n,m=0}^{\infty} C_{nm} X_{nm}(x, y) e^{-\lambda_{nm} t}
\end{align*}
\]

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where
\[
C_{nm} = \frac{\langle \phi, X_{nm} \rangle}{\langle X_{nm}, X_{nm} \rangle} = \frac{\int_0^k \int_0^l \phi X_{nm} \, dx \, dy}{\int_0^k \int_0^l X_{nm}^2 \, dx \, dy}.
\]

9. (10 points) Suppose \( u \in C^2(\overline{\Omega}) \) is a solution of
\[
\Delta u = f \geq 0 \quad x \in \Omega.
\]
Show that
\[
u(x) \leq \int_{\partial B(x,r)} u(y) \, dy
\]
for all \( B(x,r) \subset \Omega \).

**Answer:** Define the function
\[
\phi(r) = \int_{\partial B(x,r)} u(y) \, dS(y)
\]
for \( r > 0 \) and \( \phi(0) = u(x) \). Using the assumption that \( u \) is continuous, we conclude that \( \phi \) is continuous. We now look at \( \phi'(r) \).

\[
\phi'(r) = \frac{d}{dr} \int_{\partial B(x,r)} u(y) \, dS(y)
\]
\[
= \frac{d}{dr} \int_{\partial B(0,1)} u(x + rz) \, dS(z)
\]
\[
= \int_{\partial B(0,1)} \nabla u(x + rz) \cdot z \, dS(z)
\]
\[
= \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r} \, dS(y)
\]
\[
= \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \, dS(y)
\]
\[
= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \, dS(y)
\]
\[
= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u \, dy
\]
\[
\geq 0.
\]

Therefore, \( \phi' \) is an increasing function of \( r \). Therefore, we conclude that \( \phi(0) \leq \phi(r) \).

Therefore, we have
\[
u(x) \leq \int_{\partial B(x,r)} u(y) \, dS(y),
\]
as claimed.
10. (8 points) Find the smooth function $f$ which yields the best lower bound for $\int_0^1 (g'(x))^2 \, dx$ among functions satisfying $g(0) = 3$, $g(1) = 4$.

**Answer:** By Dirichlet’s principle, we know that the function which minimizes $I(w) = \int_{\Omega} |\nabla w|^2 \, dx$ subject to certain boundary conditions is the harmonic function on $\Omega$ which satisfies those boundary conditions. Therefore, we look for the solution to

$$
\begin{cases}
  g'' = 0 & \text{for } 0 < x < 1 \\
  g(0) = 3, g(1) = 4.
\end{cases}
$$

Clearly, the harmonic functions on an interval are linear functions. That is, $g(x) = A + Bx$.

The boundary condition $g(0) = 3 \implies A = 3$.

The boundary condition $g(1) = 4 \implies B = 1$.

Therefore, the solution is $g(x) = 3 + x$. 

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