Math 220B **Final Exam** **March 18, 2002**

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Name: ____________________

Please sign below in acknowledgment and acceptance of the Honor Code.

Signature: ____________________

This exam is closed notes, closed book. The exam is worth a total of 100 points. The point value of each problem is indicated. Please show all work and clearly mark your answer.

Number	Points
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
Total	

1. (12 points)

(a) (6 points) Find the Green's function for the tilted half-plane

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 > 0\}.$$

For $x = (x_1, x_2), y = (y_1, y_2) \in \Omega$, express your Green's function $G(x, y)$ in terms of x_1, x_2, y_1 and y_2.

(b) (6 points) Use the Green's function from part (a) to write the solution formula for

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

Simplify your answer as much as possible.

2. (8 points) Find a Neumann function for the interval $[a, b] \subset \mathbb{R}$. That is, find a function $N(x, y)$ such that for each $x \in (a, b)$,

$$\begin{cases} -\Delta_y N(x,y) = \delta_x & a < y < b \\ \frac{\partial N}{\partial \nu}(x,y) = -\frac{1}{2} & y = a, b \end{cases}$$

3. (10 points) Let Ω be the triangle bounded by $x_2 = 0$, $x_2 = 1 - x_1$ and $x_2 = 1 + x_1$. Let $\lambda_i(\Omega)$ be the i^{th} eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

(a) Prove that

$$\frac{5}{4}\pi^2 \leq \lambda_1(\Omega).$$

(b) Prove that

$$\frac{5}{2}\pi^2 \leq \lambda_2(\Omega).$$

4. (12 points) Consider the eigenvalue problem with Neumann boundary conditions,

$$(*) \begin{cases} -\Delta u = \lambda u & x \in \Omega \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases}$$

(a) (8 points) Let $X_n \equiv \{w \in C^2(\Omega); w \not\equiv 0, \langle w, v_i \rangle = 0 \text{ for } i = 1, \ldots, n-1\}$ where the v_i are the first $n-1$ eigenfunctions. Prove that the nth eigenvalue of (*) satisfies

$$\lambda_n = \min_{w \in X_n} \frac{||\nabla w||_{L^2}^2}{||w||_{L^2}^2}.$$

(b) (4 points) Use the results of part (a) to give an estimate on the second eigenvalue for (*) (eigenvalue problem of the Laplacian with Neumann boundary conditions) in the case when Ω is the triangle from the previous problem (the triangle bounded by $x_2 = 0$, $x_2 = 1 - x_1$ and $x_2 = 1 + x_1$).

5. (15 points) Determine whether the following statements are true or false. **Briefly explain your answer.**

(a) Let Ω be an open, bounded, connected subset of $\mathbb{R}^n$. Consider the exterior Neumann problem,

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega. \end{cases}$$

A necessary condition for solvability is

$$\int_{\partial\Omega} g(y)\, dS(y) = 0.$$

(b) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $a(x) \geq 0$. All eigenvalues of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ \frac{\partial u}{\partial \nu} + a(x)u = 0 & x \in \partial\Omega \end{cases}$$

are non-negative.

(c) Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < \pi, 0 < x_2 < \pi\}$. Suppose u and v are linearly independent eigenfunctions of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then u and v are orthogonal.

(d) Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < \pi\}$. Suppose u and v are linearly independent eigenfunctions of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then u and v are orthogonal.

(e) Let Ω be an open, bounded set in $\mathbb{R}^n$. Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function on Ω. If there exists a point $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u(x),$$

then $u(x) \equiv$ constant.

6. (7 points) Answer the following short answer questions.

(a) (4 points) State the minimax principle for the nth eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \subset \mathbb{R}^n \\ u = 0 & x \in \partial\Omega \end{cases}$$

(b) (3 points) State Liouville's Theorem.

7. (10 points) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $\{v_i, \lambda_i\}$ be the eigenfunctions and eigenvalues of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

Solve the following initial/boundary value problem,

$$\begin{cases} u_t - k\Delta u = 0 & x \in \Omega \\ u(x,0) = C_1 v_1(x) + C_2 v_2(x) & \\ u(x,t) = 0 & x \in \partial\Omega. \end{cases}$$

8. (16 points)

(a) (4 points) Let $B_3(0,1)$ be the ball of radius 1 about the origin in $\mathbb{R}^3$. Use the fundamental solution of Laplace's equation to construct a solution of

$$(*) \begin{cases} \Delta u = 0 & x \in (B_3(0,1))^c \\ u = 1 & x \in \partial B_3(0,1) \end{cases}$$

which decays to zero as $|x| \to +\infty$.

(b) (4 points) Prove uniqueness of solutions of (*) which decay to zero as $|x| \to +\infty$.

(c) (4 points) Give the definition of a double-layer potential with moment h. If we can write the solution of (*) as a double-layer potential, what equation must h satisfy?

(d) (4 points) Using parts (a)-(c), explain why we *cannot* write the solution of (*) as a double-layer potential.

9. (10 points) Let

$$f_n(x) = \sqrt{\frac{n}{4\pi}} e^{-nx^2/4}.$$

Prove that f_n converges weakly to δ_0 in the sense of distributions as $n \to +\infty$.

Scratch Paper