Math 220B **Final Exam Solutions** **March 18, 2002**

1. (12 points)

   (a) (6 points) Find the Green's function for the tilted half-plane

   $$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 > 0\}.$$

   For $x = (x_1, x_2), y = (y_1, y_2) \in \Omega$, express your Green's function $G(x, y)$ in terms of x_1, x_2, y_1 and y_2

   Answer:

   $$\boxed{G(x,y) = -\frac{1}{2\pi} \ln|(y_1, y_2) - (x_1, x_2)| + \frac{1}{2\pi} \ln|(y_1, y_2) + (x_2, x_1)|}$$

   (b) (6 points) Use the Green's function from part (a) to write the solution formula for

   $$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

   Simplify your answer as much as possible.

   Answer:

   $$\boxed{u(x) = \frac{x_1 + x_2}{\pi\sqrt{2}} \int_{\partial\Omega} \frac{g(y)}{|y - x|^2}\, dS(y).}$$

2. (8 points) Find a Neumann function for the interval $[a, b] \subset \mathbb{R}$. That is, find a function $N(x, y)$ such that for each $x \in (a, b)$,

   $$\begin{cases} -\Delta_y N(x,y) = \delta_x & a < y < b \\ \frac{\partial N}{\partial \nu}(x,y) = -\frac{1}{2} & y = a, b \end{cases}$$

   Answer:

   $$\boxed{N(x,y) = -\frac{1}{2}|y - x|.}$$

3. (10 points) Let Ω be the triangle bounded by $x_2 = 0$, $x_2 = 1 - x_1$ and $x_2 = 1 + x_1$. Let $\lambda_i(\Omega)$ be the i^{th} eigenvalue of

   $$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

   (a) Prove that

   $$\frac{5}{4}\pi^2 \le \lambda_1(\Omega).$$

   Answer: Let $\Omega_1 = \{(x_1, x_2) : -1 < x_1 < 1, 0 < x_2 < 1\}$. We know $\Omega \subset \Omega_1 \implies \lambda_i(\Omega) \ge \lambda_i(\Omega_1)$. Now

   $$\lambda_i(\Omega_1) = \left(\frac{n\pi}{2}\right)^2 + (m\pi)^2.$$

   Therefore,

   $$\lambda_1(\Omega_1) = \frac{5}{4}\pi^2 \le \lambda_1(\Omega).$$

(b) Prove that

$$\frac{5}{2}\pi^2 \le \lambda_2(\Omega).$$

Answer: Let Ω_2 be the square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. This square has side lengths $\sqrt{2}$, and contains Ω. The eigenvalues of Ω_2 are given by

$$\lambda_i(\Omega_2) = \left(\frac{n\pi}{\sqrt{2}}\right)^2 + \left(\frac{m\pi}{\sqrt{2}}\right)^2.$$

Therefore,

$$\lambda_2(\Omega_2) = \frac{\pi^2}{2} + 2\pi^2 = \frac{5}{2}\pi^2 \le \lambda_2(\Omega).$$

4. (12 points) Consider the eigenvalue problem with Neumann boundary conditions,

$$(*) \begin{cases} -\Delta u = \lambda u & x \in \Omega \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases}$$

(a) (8 points) Let $X_n \equiv \{w \in C^2(\Omega); w \not\equiv 0, \langle w, v_i \rangle = 0 \text{ for } i = 1, \ldots, n-1\}$ where the v_i are the first $n-1$ eigenfunctions. Prove that the nth eigenvalue of (*) satisfies

$$\lambda_n = \min_{w \in X_n} \frac{||\nabla w||_{L^2}^2}{||w||_{L^2}^2}.$$

Answer: Suppose $u \in X_n$ is the minimizer of this quotient over all $w \in X_n$. Pick any $v \in X_n$ and let

$$i(t) = \frac{||\nabla(u + tv)||_{L^2(\Omega)}^2}{||u + tv||_{L^2}^2}.$$

If u is a minimizer, then i has a local minimum at $t = 0$, and, therefore, $i'(0) = 0$. By a straightforward calculation, we see that

$$i'(0) = \frac{(\int u^2)(\int 2\nabla u \cdot \nabla v) - (\int |\nabla u|^2)(\int 2uv)}{(\int u^2)^2}.$$

We see that $i'(0) = 0 \implies$

$$-\int_\Omega \Delta uv\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,dS(x) = m \int_\Omega uv\,dx.$$

Therefore,

$$\int_\Omega [\Delta u + mu] v\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,dS(x)$$

for all $v \in X_n$.

Now let v_j be one of the first $n-1$ eigenfunctions for this problem. By assumption, $u \in X_n$ implies that u is orthogonal to v_j. Therefore, we see that

$$\begin{aligned}
\int_\Omega [\Delta u + mu] v_j \, dx &= \int_\Omega \Delta u v_j \, dx \\
&= \int_\Omega u \Delta v_j \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v_j \, dS(x) - \int_{\partial\Omega} u \frac{\partial v_j}{\partial \nu} \, dS(x) \\
&= -\lambda_j \int_\Omega u v_j \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v_j \, dS(x). \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v_j \, dS(x).
\end{aligned}$$

Now let h be an arbitrary trial function (C^2 function which vanishes on the boundary of Ω). Define

$$v = h - \sum_{i=1}^{n-1} c_i v_i$$

where

$$c_i \equiv \frac{\langle h, v_i \rangle}{\langle v_i, v_i \rangle}.$$

First, we note that

$$\langle v, v_j \rangle = \langle h - \sum_{i=1}^{n-1} c_i v_i \rangle = 0.$$

Therefore, we conclude that $v \in X_n$.
Therefore,

$$\begin{aligned}
\int_\Omega [\Delta u + mu] h \, dx &= \int_\Omega [\Delta u + mu] \left\{ v + \sum_{i=1}^{n-1} c_i v_i \right\} \, dx \\
&= \int_\Omega [\Delta u + mu] v \, dx + \sum_{i=1}^{n-1} c_i \int_\Omega v_i \, dx \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS(x) + \sum_{i=1}^{n-1} c_i \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v_i \, dS(x) \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \nu} h \, dS(x) = 0,
\end{aligned}$$

using the assumption that h vanishes on $\partial\Omega$. Therefore, we conclude that

$$\Delta u + mu = 0,$$

and, thus, u is an eigenfunction with corresponding eigenvalue m. Next, we will show that m is the n^{th} eigenvalue. First, we note that $X_n \subset X_{n-1} \subset X_{n-2} \subset \ldots$. Therefore, $m \geq \lambda_{n-1} \geq \lambda_{n-2} \geq \ldots$. To show that $m \leq \lambda_n$, let v_j be an eigenfunction with corresponding eigenvalue λ_j for $j \geq n$. Now $v_j \in X_n$. Therefore,

$$m = \min_{w \in X_n} \frac{||\nabla w||^2_{L^2(\Omega)}}{||w||^2_{L^2(\Omega)}} \leq \frac{||\nabla v_j||^2_{L^2(\Omega)}}{||v_j||_{L^2(\Omega)^2}} = \frac{\int_\Omega |\nabla v_j|^2}{\int_\Omega v_j^2} = -\frac{\int_\Omega v_j \Delta v_j}{\int_\Omega v_j^2} = \frac{\lambda_j \int_\Omega v_j^2}{\int_\Omega v_j^2} = \lambda_j.$$

(b) (4 points) Use the results of part (a) to give an estimate on the second eigenvalue for (*) (eigenvalue problem of the Laplacian with Neumann boundary conditions) in the case when Ω is the triangle from the previous problem (the triangle bounded by $x_2 = 0$, $x_2 = 1 - x_1$ and $x_2 = 1 + x_1$).

Answer: We need to find a function in X_2, the space of functions which are orthogonal to the first eigenfunction. As the first eigenvalue for the Laplacian with Neumann boundary conditions is zero, we know the first eigenfunction is the constant function. Therefore, X_2 consists of C^2 functions which are orthogonal to constant functions; that is, functions which satisfy

$$\int_\Omega v\, dx = 0.$$

Here Ω is the triangle with vertices at $(1,0)$, $(0,1)$ and $(-1,0)$. One function in X_2 would be $v(x,y) = x$. Using this test function, we get the following approximation for the second eigenvalue,

$$\lambda_2 \approx \frac{\int_\Omega dx_1\, dx_2}{\int_\Omega x_1^2\, dx_1\, dx_2} = \frac{1}{1/6} = 6.$$

5. (15 points) Determine whether the following statements are true or false. **Briefly explain your answer.**

(a) Let Ω be an open, bounded, connected subset of $\mathbb{R}^n$. Consider the exterior Neumann problem,

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega. \end{cases}$$

A necessary condition for solvability is

$$\int_{\partial\Omega} g(y)\, dS(y) = 0.$$

Answer: False. For example, let $\Omega = B(0,1)$ in $\mathbb{R}^3$ and consider $u(x) = \frac{1}{|x|}$. We see that u is a solution of

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ \dfrac{\partial u}{\partial \nu} = -1 & x \in \partial\Omega \end{cases}$$

but $\int_{\partial\Omega} -1\, dS(x) \neq 0$.

(b) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $a(x) \geq 0$. All eigenvalues of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ \frac{\partial u}{\partial \nu} + a(x)u = 0 & x \in \partial\Omega \end{cases}$$

are non-negative.

Answer: $\boxed{\text{True.}}$

$$\begin{aligned} \lambda \int_\Omega u^2\,dx = -\int_\Omega u\Delta u\,dx &= \int_\Omega |\nabla u|^2\,dx - \int_\Omega u\frac{\partial u}{\partial \nu}\,dS(x) \\ &= \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} a(x)u^2\,dx \geq 0. \end{aligned}$$

Therefore, $\lambda \geq 0$.

(c) Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < \pi, 0 < x_2 < \pi\}$. Suppose u and v are linearly independent eigenfunctions of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then u and v are orthogonal.

Answer: $\boxed{\text{False.}}$ The eigenvalues are given by $\lambda_{mn} = m^2 + n^2$. We see that $\lambda = 5$ is an eigenvalue with multiplicity 2. Therefore, it has a two-dimensional eigenspace, and consequently the eigenfunctions need not be orthogonal.

(d) Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < \pi\}$. Suppose u and v are linearly independent eigenfunctions of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then u and v are orthogonal.

Answer: $\boxed{\text{True.}}$ The eigenvalues are given by $\lambda_{mn} = (m\pi)^2 + n^2$. All eigenvalues are distinct. We know that for symmetric boundary conditions, eigenfunctions corresponding to distinct eigenvalues are orthogonal. Therefore, all linearly independent eigenfunctions must be orthogonal.

(e) Let Ω be an open, bounded set in $\mathbb{R}^n$. Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function on Ω. If there exists a point $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u(x),$$

then $u(x) \equiv$ constant.

Answer: $\boxed{\text{False.}}$ We need the domain Ω to be connected for the strong maximum principle to hold.

6. (7 points) Answer the following short answer questions.

(a) (4 points) State the minimax principle for the nth eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \subset \mathbb{R}^n \\ u = 0 & x \in \partial\Omega \end{cases}$$

Answer: Let $w_1, \ldots, w_n$ be a set of n linearly independent trial functions (C^2 functions which vanish on $\partial\Omega$). Let

$$\lambda_n^*(w_1, \ldots, w_n) = \max_{c \neq 0} \left\{ \frac{||\nabla w||^2_{L^2(\Omega)}}{||w||^2_{L^2(\Omega)}} : w = \sum_{i=1}^{n} c_i w_i \right\}.$$

Then

$$\lambda_n = \min \lambda_n^*(w_1, \ldots, w_n),$$

where the minimum is taken over all possible sets of n linearly independent trial functions.

(b) (3 points) State Liouville's Theorem.

Answer: If u is a bounded, harmonic function on $\mathbb{R}^n$, then u must be constant.

7. (10 points) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $\{v_i, \lambda_i\}$ be the eigenfunctions and eigenvalues of

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

Solve the following initial/boundary value problem,

$$\begin{cases} u_t - k\Delta u = 0 & x \in \Omega \\ u(x,0) = C_1 v_1(x) + C_2 v_2(x) & \\ u(x,t) = 0 & x \in \partial\Omega. \end{cases}$$

Answer: Look for a solution of the form $u(x,t) = X(x)T(t)$. Then we see that

$$u_n(x,t) = A_n v_n e^{-k\lambda_n t}$$

is a solution of the heat equation which satisfies the boundary condition for each n. Now, we want

$$u(x,0) = C_1 v_1(x) + C_2 v_2(x).$$

Therefore, by letting $A_1 = C_1$, $A_2 = C_2$ and $A_i = 0$ for $i \geq 3$, we arrive at the solution

$$\boxed{u(x,t) = C_1 v_1(x) e^{-k\lambda_1 t} + C_2 v_2(x) e^{-k\lambda_2 t}.}$$

8. (16 points)

(a) (4 points) Let $B_3(0,1)$ be the ball of radius 1 about the origin in $\mathbb{R}^3$. Use the fundamental solution of Laplace's equation to construct a solution of

$$(*)\begin{cases} \Delta u = 0 & x \in (B_3(0,1))^c \\ u = 1 & x \in \partial B_3(0,1) \end{cases}$$

which decays to zero as $|x| \to +\infty$.

Answer:

$$\boxed{u(x) = \frac{1}{|x|}.}$$

(b) (4 points) Prove uniqueness of solutions of (*) which decay to zero as $|x| \to +\infty$.

Answer: Suppose u and v are two solutions of (*) which decay to zero as $|x| \to +\infty$. Fix $C > 1$. Let $\Omega_C = (B_3(0,1))^c \cap B_3(0,C)$. Then u and v are harmonic on Ω_C, and $w = u - v$ satisfies

$$\begin{cases} \Delta w = 0 & x \in \Omega_C \\ w = 0 & x \in \partial B_3(0,1) \\ w \leq \epsilon & x \in \partial B_3(0,C) \end{cases}$$

for some $\epsilon = \epsilon(C)$. By the maximum principle,

$$\max_{\overline{\Omega}_C} w(x) = \max_{\partial\Omega_C} w(x) \leq \max\{0, \epsilon\}.$$

Therefore, $w \leq \max\{0, \epsilon\}$. This is true for all $\epsilon > 0$ by choosing C sufficiently large. Therefore, we conclude that $w = u - v \leq 0$ on $(B_3(0,1))^c$. By a similar analysis with $\widetilde{w} = v - u$, we conclude that $\widetilde{w} = v - u \leq 0$ on $(B_3(0,1))^c$. Therefore, we conclude that $u = v$.

(c) (4 points) Give the definition of a double-layer potential with moment h. If we can write the solution of (*) as a double-layer potential, what equation must h satisfy?

Answer:

$$\boxed{u(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu_y}(x - y)\, dS(y)}$$

where

$$\boxed{g(x) = -\frac{1}{2} h(x) - \int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu_y}(x - y)\, dS(y).}$$

(d) (4 points) Using parts (a)-(c), explain why we *cannot* write the solution of (*) as a double-layer potential.

Answer: Suppose we can write the solution as a double-layer potential. By parts (a) and (b), we know the unique solution (which decays to zero) is given by $u(x) = \frac{1}{|x|}$. (Note: A solution given by the double-layer potential will decay to zero.) By part (c), we know that if the solution is given by a double-layer potential, then there must exist a continuous function h such that

$$\frac{1}{|x|} = -\int_{\partial B_3(0,1)} h(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y)$$

which satisfies

$$1 = -\frac{1}{2}h(x) + \frac{1}{|x|}$$

for all $x \in \partial B_3(0,1)$. Therefore, we need

$$1 = -\frac{1}{2}h(x) + 1,$$

which implies $h = 0$. But, we can see this cannot be satisfied.

9. (10 points) Let

$$f_n(x) = \sqrt{\frac{n}{4\pi}} e^{-nx^2/4}.$$

Prove that f_n converges weakly to δ_0 in the sense of distributions as $n \to +\infty$.

Answer: Let F_{f_n} be the distribution such that

$$(F_{f_n}, \phi) \equiv \int_{-\infty}^{\infty} f_n(x)\phi(x)\, dx.$$

We need to show that

$$(F_{f_n}, \phi) \to (\delta_0, \phi) = \phi(0)$$

as $n \to +\infty$. That is, we need to show that for all $\epsilon > 0$, there exists an N such that

$$|(F_{f_n}, \phi) - \phi(0)| < \epsilon$$

for $n \geq N$. Using the fact that

$$\int_{-\infty}^{\infty} \sqrt{\frac{n}{4\pi}} e^{-nx^2/4}\, dx = 1,$$

we write

$$\begin{aligned}
|(F_{f_n}, \phi) - \phi(0)| &= \left| \int_{-\infty}^{\infty} \sqrt{\frac{n}{4\pi}} e^{-nx^2/4} [\phi(x) - \phi(0)]\, dx \right| \\
&\leq \left| \int_{B(0,\delta)} \sqrt{\frac{n}{4\pi}} e^{-nx^2/4} [\phi(x) - \phi(0)]\, dx \right| \\
&\quad + \left| \int_{\mathbb{R} - B(0,\delta)} \sqrt{\frac{n}{4\pi}} e^{-nx^2/4} [\phi(x) - \phi(0)]\, dx \right| \\
&= I + J
\end{aligned}$$

for δ to be determined below.

Now for term I, we bound as follows. We write

$$\begin{aligned}|I| &= \left| \int_{B(0,\delta)} \sqrt{\frac{n}{4\pi}} e^{-nx^2/4} [\phi(x) - \phi(0)]\, dx \right| \\ &\leq |\phi(x) - \phi(0)|_{L^\infty(B(0,\delta))} < \frac{\epsilon}{2}\end{aligned}$$

by choosing δ sufficiently small, using the fact that ϕ is continuous.

Now for that choice of δ, term J is bounded as follows. We write

$$\begin{aligned}|J| &= \left| \int_{\mathbb{R}-B(0,\delta)} \sqrt{\frac{n}{4\pi}} e^{-nx^2/4} [\phi(x) - \phi(0)]\, dx \right| \\ &\leq C\sqrt{\frac{n}{4\pi}} e^{-n\delta^2/8} \int_{\mathbb{R}-B(0,\delta)} e^{-nx^2/8}\, dx \\ &\leq C\sqrt{\frac{n}{4\pi}} e^{-n\delta^2/8} < \frac{\epsilon}{2}\end{aligned}$$

by choosing n sufficiently large. Therefore, our claim is proven.