Math 220B Final Exam March 19, 2003

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Name: ______________________________

Please sign below in acknowledgment and acceptance of the Honor Code.

Signature: ______________________________

This exam is closed notes, closed book. The exam is worth a total of 116 points. The point value of each problem is indicated. Please show all work and clearly mark your answer.

Number	Points
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	
Total	

1. (14 points) Let Ω be the upper half of the unit disk in $\mathbb{R}^2$. That is, let

$$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y > 0\}.$$

Use separation of variables to solve

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ u(r,0) = 0 = u(r,\pi) & \\ u(1,\theta) = \theta(\theta - \pi). & \end{cases}$$

You do **not** need to evaluate any integrals.

2. (10 points) Let Ω be the upper half-plane in $\mathbb{R}^2$. That is, let

$$\Omega = \{(x, y) \in \mathbb{R}^2, y > 0\}.$$

Consider the boundary-value problem,

$$\begin{cases} \Delta u = 0 & (x, y) \in \Omega \\ u = 0 & (x, y) \in \partial\Omega. \end{cases} \tag{1}$$

(a) Prove uniqueness of *bounded* solutions of (1). *Hint:* Suppose u is a solution of (1). Consider the odd reflection of u across the x-axis; that is, consider the function v defined as

$$v(x, y) = \begin{cases} u(x, y) & y > 0 \\ -u(x, -y) & y < 0. \end{cases}$$

(b) Give an unbounded counterexample.

3. (18 points) Let Ω be the triangle with vertices at $(1,0)$, $(-1,0)$ and $(0,2)$. Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{2}$$

(a) Use the Comparison Principle to get an upper bound on the first eigenvalue for this eigenvalue problem. In particular, find the best upper bound on λ_1 among all rectangles contained within Ω with sides parallel to the coordinate axes.

(b) Let w_1, w_2 be two C^2 functions which are identically zero for $(x, y) \in \partial\Omega$. Explain how to use the Rayleigh-Ritz method with w_1 and w_2 to approximate the second eigenvalue of (2).

(c) Find two linearly independent functions w_1 and w_2 which can be used for this approximation. (You do **not** need to apply the Rayleigh-Ritz approximation to these functions.)

4. (8 points) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Consider

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} + u = g & x \in \partial\Omega. \end{cases} \tag{3}$$

(a) State the definition of a single-layer potential with moment h.

(b) In order to write the solution of (3) as a single-layer potential with moment h, what integral equation must h satisfy?

5. (10 points) Recall that a Neumann function satisfies

$$\begin{cases} -\Delta_y N(x,y) = \delta_x & y \in \Omega \subset \mathbb{R}^n \\ \frac{\partial N}{\partial \nu}(x,y) = C & y \in \partial\Omega \end{cases}$$

for each $x \in \Omega$, where $C = \frac{1}{\int_{\partial\Omega} dS}$. Find the Neumann function for the first quadrant

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}.$$

6. (8 points) Let $F_n : \mathcal{D} \to \mathbb{R}$ be the distribution defined such that

$$(F_n, \phi) = \int_{-\infty}^{\infty} \sin(nx)\phi(x)\,dx \qquad \forall \phi \in \mathcal{D}.$$

Show that F_n converges to 0 weakly as $n \to +\infty$.

7. (8 points) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Assume $a(x) > 0$ for all $x \in \partial\Omega$. Consider the eigenvalue problem

$$\begin{cases} -\Delta X = \lambda X & x \in \Omega \\ \frac{\partial X}{\partial \nu} + a(x)X = 0 & x \in \partial\Omega. \end{cases}$$

Prove that all eigenvalues are positive.

8. (10 points) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Suppose $u \in C^2(\overline{\Omega})$ is a solution of

$$\Delta u = f \geq 0 \qquad x \in \Omega.$$

Show that

$$u(x) \leq \fint_{\partial B(x,r)} u(y)\, dS(y)$$

for all $B(x,r) \subset \Omega$.

9. (18 points) Determine whether the following statements are true or false. **Provide a reason for your answer.**

   (a) Let h be a continuous function. The function

   $$\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)\,dS(y)$$

   is harmonic for all $x \in \mathbb{R}^n$.

   (b) Let h be a continuous function. The function

   $$\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_y}(x-y)\,dS(y)$$

   is continuous for all $x \in \mathbb{R}^n$.

(c) Let h be a continuous function. Let $n \geq 2$. The function

$$\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y)$$

is $O(|x|^{2-n})$.

(d) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Assume $a(x) \not\equiv 0$. There exists at most one solution of

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} - a(x)u = 0 & x \in \partial\Omega. \end{cases}$$

(e) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a bounded function. There exists at most one bounded solution u of

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

(f) If u is a harmonic function on the rectangle

$$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\},$$

then

$$\int_0^a u_y(x,0)\,dx + \int_0^b u_x(a,y)\,dy + \int_0^a u_y(x,b)\,dx + \int_0^b u_x(0,y)\,dy = 0.$$

10. (12 points) Consider the initial/boundary value problem

$$\begin{cases} u_t - u_{xx} + u_x = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) & 0 < x < l \\ u_x(0,t) = 0 = u_x(l,t). \end{cases} \tag{4}$$

(a) Suppose u is a solution of this problem. Find a function g such that the function $v = gu$ satisfies

$$v_t - v_{xx} + v = 0.$$

Write the initial/boundary value problem that v satisfies.

(b) In part (a) you show that $v = gu$ will satisfy an initial/boundary value problem of the form

$$\begin{cases} v_t - v_{xx} + v = 0 & 0 < x < l, t > 0 \\ v(x,0) = f(x) & 0 < x < l \\ v \text{ satisfies symmetric B.C.s} & \end{cases} \tag{5}$$

Suppose λ_n, X_n are the eigenvalues and corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X \text{ satisfies } (*) & x = 0, l, \end{cases}$$

where (*) denotes the symmetric boundary conditions in (5). Write the solution of (5) in terms of X_n, λ_n and f.

Scratch Paper