**Math 220B** **Final Exam Solutions** **March 19, 2003**

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**Name:** ______________________________

Please sign below in acknowledgment and acceptance of the Honor Code.

**Signature:** ______________________________

This exam is closed notes, closed book. The exam is worth a total of 116 points. The point value of each problem is indicated. Please show all work and clearly mark your answer.

| Number | Points |
|---|---|
| 1. | |
| 2. | |
| 3. | |
| 4. | |
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| Total | |

1. (14 points) Let $\Omega$ be the upper half of the unit disk in $\mathbb{R}^2$. That is, let

$$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y > 0\}.$$

Use separation of variables to solve

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ u(r,0) = 0 = u(r,\pi) \\ u(1,\theta) = \theta(\theta - \pi). \end{cases}$$

You do **not** need to evaluate any integrals.

**Answer:** First, we write the equation in polar coordinates,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Now, using separation of variables, we look for a solution of the form $u(r,\theta) = R(r)\Theta(\theta)$. Plugging this into our equation, we have

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Dividing by $R\Theta$ and multiplying by $r^2$, we get

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Our boundary condition

$$u(0,\theta) = 0 = u(r,\theta)$$

leads us to the eigenvalue problem

$$\begin{cases} -\Theta'' = \lambda\Theta & 0 < \theta < \pi \\ \Theta(0) = 0 = \Theta(\pi). \end{cases}$$

We know the solutions of this eigenvalue problem are

$$\Theta_n(\theta) = \sin(n\theta) \qquad \lambda_n = n^2, n = 1, 2, \ldots$$

Now we look at our equation for $R_n$, for $n = 1, 2, \ldots$. We know

$$r^2 R_n'' + rR_n' - n^2 R_n = 0$$

has solutions

$$R_n = C_n r^n + D_n r^{-n} \qquad n = 1, 2, \ldots$$

As we do not want our solution to blow up as $r \to 0$, we discard the solutions $r^{-n}$. Therefore, our solutions for $R_n$ are

$$R_n(r) = C_n r^n \qquad n = 1, 2, \ldots$$

Therefore, we let

$$\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty} R_n(r)\Theta_n(\theta) \\ &= \sum_{n=1}^{\infty} C_n r^n \sin(n\theta). \end{aligned}$$

Our other boundary condition $u(1,\theta) = \theta(\theta-\pi)$ implies we want to find constants $C_n$ such that

$$u(1,\theta) = \sum_{n=0}^{\infty} C_n \sin(n\theta) = \theta(\theta-\pi).$$

This is the Fourier sine series for our boundary data. We know our coefficients $C_n$ must be given by

$$\begin{aligned} C_n &= \frac{\langle \theta(\theta-\pi), \sin(n\theta)\rangle}{\langle \sin(n\theta), \sin(n\theta)\rangle} \\ &= \frac{\int_0^\pi \theta(\theta-\pi)\sin(n\theta)\,d\theta}{\int_0^\pi \sin^2(n\theta)\,d\theta}. \end{aligned}$$

Therefore, our solution is given by

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} C_n r^n \sin(n\theta)}$$

where

$$\boxed{C_n = \frac{\int_0^\pi \theta(\theta-\pi)\sin(n\theta)\,d\theta}{\int_0^\pi \sin^2(n\theta)\,d\theta}.}$$

2. (10 points) Let $\Omega$ be the upper half-plane in $\mathbb{R}^2$. That is, let

$$\Omega = \{(x,y) \in \mathbb{R}^2, y > 0\}.$$

Consider the boundary-value problem,

$$\begin{cases} \Delta u = 0 & (x,y) \in \Omega \\ u = 0 & (x,y) \in \partial\Omega. \end{cases} \tag{1}$$

(a) Prove uniqueness of *bounded* solutions of (1). *Hint:* Suppose $u$ is a solution of (1). Consider the odd reflection of $u$ across the $x$-axis; that is, consider the function $v$ defined as

$$v(x,y) = \begin{cases} u(x,y) & y > 0 \\ -u(x,-y) & y < 0. \end{cases}$$

**Answer:** Suppose $u$ is a solution of (1). Then by defining $v$ as the odd reflection of $u$, we note that $v$ is a harmonic function in all of $\mathbb{R}^2$. Also, by assumption, $u$ is bounded. Therefore, $v$ is bounded. By Liouville's Theorem, the only bounded, harmonic functions in all of $\mathbb{R}^n$ are constant functions. Therefore, $v$ must be constant. But, $v(x,0) = 0$. Therefore, $v(x,y) \equiv 0$, which implies $u(x,y) \equiv 0$.

(b) Give an unbounded counterexample.

**Answer:** We note that $u(x,y) = Cy$ is a solution of (1) for any $C \in \mathbb{R}$. Therefore, for any $C_1 \neq C_2$ (neither of which is zero), $u_1(x,y) = C_1 y$ and $u_2(x,y) = C_2 y$ are two unbounded solutions of (1).

3. (18 points) Let $\Omega$ be the triangle with vertices at $(1,0)$, $(-1,0)$ and $(0,2)$. Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{2}$$

(a) Use the Comparison Principle to get an upper bound on the first eigenvalue for this eigenvalue problem. In particular, find the best upper bound on $\lambda_1$ among all rectangles contained within $\Omega$ with sides parallel to the coordinate axes.

**Answer:** For any rectangle $R$ contained within $\Omega$ with sides parallel to the axes, and vertices $(x,0), (-x,0), (x,y), (-x,y)$, the eigenvalues of $R$ are given by

$$\lambda_{nm}(R) = \left(\frac{n\pi}{2x}\right)^2 + \left(\frac{m\pi}{y}\right)^2.$$

For any rectangle contained within $\Omega$, we can get a better estimate if we extend the rectangle so that its vertices intersect the boundary. Therefore, we want to minimize the first eigenvalue for all rectangles contained within $\Omega$ such that $y = 2 - 2x$. That is, we want to minimize the function

$$f(x) = \frac{1}{(2x)^2} + \frac{1}{y(x)^2} = \frac{1}{4x^2} + \frac{1}{(2-2x)^2} = \frac{1}{4}\left[\frac{1}{x^2} + \frac{1}{(1-x)^2}\right].$$

For simplicity, we neglect the coefficient $1/4$. We just need to minimize

$$g(x) = \frac{1}{x^2} + \frac{1}{(1-x)^2}.$$

We look for critical points.

$$g'(x) = \frac{-2}{x^3} + \frac{2}{(1-x)^3}.$$

Now $g'(x) = 0$ implies

$$\frac{1}{(1-x)^3} = \frac{1}{x^3}$$

or

$$x^3 = (1-x)^3 \implies x = 1 - x \implies x = \frac{1}{2}.$$

We see that $x = 1/2$ minimizes $g$. Therefore, the best upper bound on the first eigenvalue of $\Omega$ given by rectangles with sides parallel to the coordinate axes is the rectangle with vertices $(1/2,0)$, $(-1/2,0)$, $(1/2,1)$, $(-1/2,1)$. Denote this rectangle by $R^*$. Therefore,

$$\boxed{\lambda_1(R^*) = 2\pi^2 \geq \lambda_1(\Omega).}$$

(b) Let $w_1, w_2$ be two $C^2$ functions which are identically zero for $(x, y) \in \partial\Omega$. Explain how to use the Rayleigh-Ritz method with $w_1$ and $w_2$ to approximate the second eigenvalue of (2).

**Answer:** Define the $2 \times 2$ matrices

$$A = (\langle \nabla w_j, \nabla w_k \rangle)$$

$$B = (\langle w_j, w_k \rangle).$$

Consider the equation

$$\det(A - \lambda B) = 0.$$

The larger of the roots of this second-order equation gives an approximation for the second eigenvalue of (2).

(c) Find two linearly independent functions $w_1$ and $w_2$ which can be used for this approximation. (You do **not** need to apply the Rayleigh-Ritz approximation to these functions.)

**Answer:** We need to find two functions $w_1$ and $w_2$, both of which vanish for $(x, y) \in \partial\Omega$. In particular, we want both functions to vanish along the lines $y = 0$, $y = -2x + 2$ and $y = 2x + 2$. Therefore, for example, we let

$$\begin{aligned} w_1(x, y) &= y(y - (-2x + 2))(y - (2x + 2)) \\ w_2(x, y) &= y^2(y - (-2x + 2))^2(y - (2x + 2))^2. \end{aligned}$$

4. (8 points) Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Consider

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} + u = g & x \in \partial\Omega. \end{cases} \tag{3}$$

(a) State the definition of a single-layer potential with moment $h$.

**Answer:**

$$\boxed{\overline{u}(x) = -\int_{\Omega} h(y)\Phi(y - x)\, dS(y).}$$

(b) In order to write the solution of (3) as a single-layer potential with moment $h$, what integral equation must $h$ satisfy?

**Answer:** Fix $x_0 \in \partial\Omega$. Let $\nu(x_0)$ be the outer unit normal to $\partial\Omega$ at $x_0$. For all $t < 0$ such that $x_0 + t\nu(x_0) \in \Omega$, let

$$i^{x_0}(t) = \nabla \overline{u}(x_0 + t\nu(x_0)) \cdot \nu(x_0).$$

We will say the boundary condition is satisfied if

$$\lim_{t \to 0^-} i^{x_0}(t) + \overline{u}(x) = g(x_0).$$

for all $x \in \Omega$, $x_0 \in \partial\Omega$. Now

$$\lim_{t \to 0^-} i^{x_0}(t) = -\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0 - y)\, dS(y).$$

In addition, we recall that a single-layer potential is continuous for all $x \in \mathbb{R}^n$. Therefore, in order to find a solution as a single-layer potential, we need $h$ to satisfy

$$\boxed{-\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_x}(x_0 - y)\, dS(y) - \int_{\partial\Omega} h(y)\Phi(y - x_0)\, dS(y) = g(x_0)}$$

for all $x_0 \in \partial\Omega$.

5. (10 points) Recall that a Neumann function satisfies

$$\begin{cases} -\Delta_y N(x, y) = \delta_x & y \in \Omega \subset \mathbb{R}^n \\ \frac{\partial N}{\partial \nu}(x, y) = C & y \in \partial\Omega \end{cases}$$

for each $x \in \Omega$, where $C = \frac{1}{\int_{\partial\Omega} dS}$. Find the Neumann function for the first quadrant

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}.$$

**Answer:** We note that $\int_{\partial\Omega} dS = 0$ for $\Omega$ the first quadrant. Therefore, our Neumann function must satisfy

$$\begin{cases} -\Delta_y N(x, y) = \delta_x & y \in \Omega \subset \mathbb{R}^n \\ \frac{\partial N}{\partial \nu}(x, y) = 0 & y \in \partial\Omega. \end{cases}$$

In particular, letting $\Phi(y)$ denote the fundamental solution of Laplace's equation in $\mathbb{R}^2$, we can write $N(x, y) = \Phi(y - x) - \widetilde{h}^x(y)$ if $h^x(y)$ is a solution of

$$\begin{cases} \Delta_y h^x(y) = 0 & y \in \Omega \\ \frac{\partial h^x(y)}{\partial \nu} = \frac{\partial \Phi(y-x)}{\partial \nu} y \in \partial\Omega. \end{cases}$$

We see that if $x = (x_1, x_2) \in \Omega$, then considering the reflected points $x^* = (x_1, -x_2)$, $\widetilde{x} = (-x_1, x_2)$ and $\widetilde{x^*} = (-x_1, -x_2)$, then

$$\boxed{N(x, y) = \Phi(y - x) + \Phi(y - x^*) + \Phi(y - \widetilde{x}) + \Phi(y - \widetilde{x^*})}$$

satisfies the necessary conditions, where

$$\boxed{\Phi(y) = -\frac{1}{2\pi} \ln |y|.}$$

6. (8 points) Let $F_n : \mathcal{D} \to \mathbb{R}$ be the distribution defined such that

$$(F_n, \phi) = \int_{-\infty}^{\infty} \sin(nx)\phi(x)\, dx \qquad \forall \phi \in \mathcal{D}.$$

Show that $F_n$ converges to 0 weakly as $n \to +\infty$.

**Answer:** To show that $F_n$ converges to 0 weakly as $n \to +\infty$, we need to show that $(F_n, \phi) \to 0$ for all $\phi \in \mathcal{D}$. Consider $\phi \in \mathcal{D}$. Therefore, $\phi$ has compact support. We note that

$$\begin{aligned}(F_n, \phi) &= \int_{-\infty}^{\infty} \sin(nx)\phi(x)\,dx \\ &= -\int_{-\infty}^{\infty} \frac{1}{n}\cos(nx)\phi'(x)\,dx + \frac{1}{n}\cos(nx)\phi'(x)\bigg|_{-\infty}^{\infty}.\end{aligned}$$

Using the fact that $\phi$ has compact support, we know that the boundary terms go to zero. Therefore, we are left with

$$\begin{aligned}|(F_n, \phi)| &\leq \left| \int_{-\infty}^{\infty} \frac{1}{n}\cos(nx)\phi'(x)\,dx \right| \\ &\leq \frac{C}{n} \to 0 \text{ as } n \to +\infty.\end{aligned}$$

7. (8 points) Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Assume $a(x) > 0$ for all $x \in \partial\Omega$. Consider the eigenvalue problem

$$\begin{cases} -\Delta X = \lambda X & x \in \Omega \\ \frac{\partial X}{\partial \nu} + a(x)X = 0 & x \in \partial\Omega. \end{cases}$$

Prove that all eigenvalues are positive.

**Answer:** Suppose $\lambda$ is an eigenvalue with eigenfunction $X$. Then

$$\begin{aligned}\lambda \int_\Omega X^2\,dx &= -\int \Delta X X\,dx \\ &= +\int |\nabla X|^2\,dx - \int_{\partial\Omega} \frac{\partial X}{\partial \nu} X\,dS(x) \\ &= \int_\Omega |\nabla X|^2\,dx + \int_{\partial\Omega} a(x)X^2\,dx \geq 0,\end{aligned}$$

using the assumption that $a(x) > 0$ for all $x \in \partial\Omega$. Therefore, we see that $\lambda \geq 0$.

It just remains to show that $\lambda \neq 0$. Suppose $\lambda = 0$. Then

$$\int_\Omega |\nabla X|^2\,dx = 0 = \int_{\partial\Omega} a(x)X^2\,dx.$$

The first equality implies that $X \equiv C$. The second equality implies $X \equiv 0$ for all $x \in \partial\Omega$. Therefore, $X \equiv 0$ for all $x \in \Omega$. But, the zero function is not an eigenfunction. Therefore, all eigenvalues are positive.

8. (10 points) Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Suppose $u \in C^2(\overline{\Omega})$ is a solution of

$$\Delta u = f \geq 0 \qquad x \in \Omega.$$

Show that

$$u(x) \leq ⨍_{\partial B(x,r)} u(y)\, dS(y)$$

for all $B(x,r) \subset \Omega$.

**Answer:** Define the function

$$\phi(r) = ⨍_{\partial B(x,r)} u(y)\, dS(y)$$

for $r > 0$ and $\phi(0) = u(x)$. Using the assumption that $u$ is continuous, we conclude that $\phi$ is continuous. We now look at $\phi'(r)$.

$$\begin{aligned}
\phi'(r) &= \frac{d}{dr} ⨍_{\partial B(x,r)} u(y)\, dS(y) \\
&= \frac{d}{dr} ⨍_{\partial B(0,1)} u(x+rz)\, dS(z) \\
&= ⨍_{\partial B(0,1)} \nabla u(x+rz) \cdot z\, dS(z) \\
&= ⨍_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\, dS(y) \\
&= ⨍_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u\, dy \\
&\geq 0.
\end{aligned}$$

Therefore, $\phi'$ is an increasing function of $r$. Therefore, we conclude that $\phi(0) \leq \phi(r)$. Therefore, we have

$$u(x) \leq ⨍_{\partial B(x,r)} u(y)\, dS(y),$$

as claimed.

9. (18 points) Determine whether the following statements are true or false. **Provide a reason for your answer.**

   (a) Let $h$ be a continuous function. The function

   $$\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)\, dS(y)$$

   is harmonic for all $x \in \mathbb{R}^n$.

   **Answer:** $\boxed{\textit{False.}}$ It is not harmonic for $x \in \partial\Omega$. (It is not even continuous for $x \in \partial\Omega$.)

(b) Let $h$ be a continuous function. The function

$$\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y)$$

is continuous for all $x \in \mathbb{R}^n$.

**Answer:** $\boxed{\textit{False.}}$ For example, let $h = 1$. By Gauss' Lemma, we know that

$$\overline{\overline{u}}(x) = \begin{cases} 1 & x \in \Omega \\ \frac{1}{2} & x \in \partial\Omega \\ 0 & x \in \Omega^c \end{cases}$$

(c) Let $h$ be a continuous function. Let $n \geq 2$. The function

$$\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y)$$

is $O(|x|^{2-n})$.

**Answer:** $\boxed{\textit{True.}}$ We note that $\Phi(x-y) = \frac{C}{|x-y|^{n-2}}$. Therefore, $\frac{\partial \Phi}{\partial \nu_y}(x-y) = C|x-y|^{n-1} \cdot \nu(y) = O(|x|^{n-1}) = O(|x|^{n-2})$.

(d) Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Assume $a(x) \not\equiv 0$. There exists at most one solution of

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} - a(x)u = 0 & x \in \partial\Omega. \end{cases}$$

**False.** Suppose $\Omega = (0,1)$. If $u$ is harmonic in $\Omega$, then $u(x) = A + Bx$. Now

$$\frac{\partial u}{\partial \nu}(0) - a(0)u(0) = -u_x(0) - a(0)u(0)$$
$$\frac{\partial u}{\partial \nu}(1) - a(1)u(1) = u_x(1) - a(1)u(1).$$

Suppose $a(0) = -1$, $a(1) = 1/2$. Then our boundary conditions read

$$-u_x(0) + u(0) = 0$$
$$u_x(1) - \frac{1}{2}u(1) = 0.$$

The first boundary condition implies

$$-B + A = 0.$$

The second boundary condition implies

$$B - \frac{1}{2}(A+B) = 0.$$

We see that there are an infinite number of solutions to this system of equations, and, therefore, an infinite number of solutions to this boundary-value problem.

(e) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a bounded function. There exists at most one bounded solution $u$ of

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

**Answer:** $\boxed{True}$ This follows from the uniqueness theorem for solutions of the heat equation on $\mathbb{R}^n$. In particular, we know there exits at most one solution $u(x,t)$ which satisfies the growth estimate

$$|u(x,t)| \leq Ae^{a|x|^2}$$

for some constants $A, a$. Therefore, in particular, there exits at most one bounded solution.

(f) If $u$ is a harmonic function on the rectangle

$$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\},$$

then

$$\int_0^a u_y(x,0)\,dx + \int_0^b u_x(a,y)\,dy + \int_0^a u_y(x,b)\,dx + \int_0^b u_x(0,y)\,dy = 0.$$

**Answer:** $\boxed{False}$ Let $u(x,y) = x$. This function is harmonic on $\Omega$. Clearly, $u_y = 0$. But, $u_x(0,y) = 1 = u_x(a,y)$. Therefore,

$$\int_0^a u_y(x,0)\,dx + \int_0^b u_x(a,y)\,dy + \int_0^a u_y(x,b)\,dx + \int_0^b u_x(0,y)\,dy = 2b \neq 0.$$

10. (12 points) Consider the initial/boundary value problem

$$\begin{cases} u_t - u_{xx} + u_x = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) & 0 < x < l \\ u_x(0,t) = 0 = u_x(l,t). \end{cases} \tag{4}$$

(a) Suppose $u$ is a solution of this problem. Find a function $g$ such that the function $v = gu$ satisfies

$$v_t - v_{xx} + v = 0.$$

Write the initial/boundary value problem that $v$ satisfies.

**Answer:** If $v = gu$, then we see that

$$\begin{aligned} v_t - v_{xx} + v &= g_t u + g u_t - g_{xx} u - 2g_x u_x - g u_{xx} + gu \\ &= g(u_t - u_{xx} + u_x) - g u_x + g_t u - g_{xx} u - 2 g_x u_x + gu \\ &= u(g_t - g_{xx} + g) - u_x(g + 2g_x). \end{aligned}$$

In order for this to equal zero, we need

$$\begin{aligned} g + 2g_x &= 0 \\ g_t - g_{xx} + g &= 0. \end{aligned}$$

The first equation implies $g(x,t) = C(t)e^{-\frac{1}{2}x}$. Plugging this into the second equation, we have

$$C'(t)e^{-\frac{1}{2}x} - \frac{1}{4}C(t)e^{-\frac{1}{2}x} + C(t)e^{-\frac{1}{2}x},$$

which implies

$$C'(t) + \frac{3}{4}C(t) = 0.$$

Therefore, $C(t) = e^{-\frac{3}{4}t}$. Therefore, we let

$$\boxed{g(x,t) = e^{-\frac{3}{4}t}e^{-\frac{1}{2}x}.}$$

Then for $v = gu$, using the fact that

$$\begin{aligned} v_x(0,t) &= g_x(0,t)u(0,t) + g(0,t)u_x(0,t) = -\frac{1}{2}e^{-\frac{3}{4}t}u(0,t) = -\frac{1}{2}v(0,t) \\ v_x(l,t) &= g_x(l,t)u(l,t) + g(l,t)u_x(l,t) = -\frac{1}{2}e^{-\frac{3}{4}t}e^{-\frac{1}{2}l}u(l,t) = -\frac{1}{2}v(l,t) \end{aligned}$$

we see that $v$ is a solution of

$$\boxed{\begin{array}{ll} v_t - v_{xx} + v = 0 & 0 < x < l \\ v(x,0) = g(x,0)u(x,0) = e^{-\frac{1}{2}x}\phi(x) & \\ v_x(0,t) + \frac{1}{2}v(0,t) = 0 & \\ v_x(l,t) + \frac{1}{2}v(l,t) = 0. & \end{array}}$$

(b) In part (a) you show that $v = gu$ will satisfy an initial/boundary value problem of the form

$$\begin{cases} v_t - v_{xx} + v = 0 & 0 < x < l, t > 0 \\ v(x,0) = f(x) & 0 < x < l \\ v \text{ satisfies symmetric B.C.s} & \end{cases} \tag{5}$$

Suppose $\lambda_n, X_n$ are the eigenvalues and corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X \text{ satisfies (*)} & x = 0, l, \end{cases}$$

where (*) denotes the symmetric boundary conditions in (5). Write the solution of (5) in terms of $X_n, \lambda_n$ and $f$.

**Answer:** We look for a solution of (5) by using separation of variables. Look for a solution of the form $v(x,t) = X(x)T(t)$. Plugging this into (5), we have

$$T'X - TX'' + TX = 0,$$

which implies

$$\frac{T'}{T} - \frac{X''}{X} + 1 = 0 \implies \frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda.$$

Now the solutions of the eigenfunction problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X \text{ satisfies } (*) & x = 0, l, \end{cases}$$

are assumed to be given by $X_n, \lambda_n$. Then we see the solutions of

$$\frac{T_n'}{T_n} + 1 = -\lambda_n$$

are given by $T_n(t) = C_n e^{-(\lambda_n+1)t}$. Therefore, our solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n X_n(x) e^{-(\lambda_n+1)t}}$$

where

$$\boxed{C_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle}.}$$