1 Introduction

In this course, we study two types of second-order equations: parabolic and elliptic. Recall the following definitions. The **order** of a partial differential equation is the order of the highest derivative appearing in the equation. Linear, second-order equations for functions of two spatial variables can be written in the following form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y) \tag{1}$$

where the coefficients $a, \ldots, f$ may be functions of x and y. We will focus on the constant-coefficient case. We say an equation of the form (1) is **parabolic** if $b^2 - 4ac = 0$. We say an equation of the form (1) is **elliptic** if $b^2 - 4ac < 0$. (If $b^2 - 4ac > 0$, the equation of the form (1) is **hyperbolic**.)

Through a change of variables all parabolic equations can be reduced to an equation of the form

$$u_{xx} + \ldots = 0,$$

where "…" represent lower-order terms. Through a change of variables all elliptic equations can be reduced to an equation of the form

$$u_{xx} + u_{yy} + \ldots = 0.$$

(*Ref: See Strauss, Sec. 1.6.*)

Consequently, we will study the following two equations: (a) the heat equation (which is parabolic) in one spatial variable,

$$u_t - ku_{xx} = 0,$$

and its higher dimensional analog,

$$u_t - k\Delta u = 0 \qquad x \in \mathbb{R}^n$$

where

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i},$$

and (b) Laplace's equation (which is elliptic) in two spatial variables,

$$u_{xx} + u_{yy} = 0$$

and its higher dimensional analog,

$$\Delta u = 0.$$

Not only do these equations represent typical parabolic and elliptic equations, but they also arise physically. We begin by studying the heat equation.