Math 220B          Midterm Exam          July 25, 2002
Julie Levandosky

Name: ____________________________________________

Please sign below in acknowledgment and acceptance of the Honor Code.

Signature: ____________________________________________

This exam is closed notes, closed book. The exam is worth a total of 60 points. The point value of each problem is indicated. Please show all work and clearly mark your answer.

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1. (8 points) Let $\phi \in L^2(\mathbb{R})$. Use the Fourier transform to solve

\[
\begin{aligned}
&\begin{aligned}
&u_t - (\sin t)u_x = 0 \\
&u(x, 0) = \phi(x). \\
\end{aligned}
\end{aligned}
\]
2. (12 points) Let $\Omega$ be the upper half of the unit disk in $\mathbb{R}^2$. That is, let

$$\Omega \equiv \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y > 0\}.$$ 

Solve

$$\begin{cases} 
  u_{xx} + u_{yy} = 0 \quad & (x, y) \in \Omega \\
  u(r, 0) = 0 = u(r, \pi) \\
  u(1, \theta) = \theta(\theta - \pi).
\end{cases}$$
3. (6 points) Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Let $\alpha > 0$. Prove uniqueness of solutions to the following problem,

$$
\begin{align*}
\Delta u - \alpha u &= f \quad x \in \Omega \\
\frac{\partial u}{\partial \nu} &= g \quad x \in \partial \Omega
\end{align*}
$$
4. (12 points) Solve the following initial/boundary value problem,

\[
\begin{align*}
& u_t - u_{xx} = 0 \quad 0 < x < \pi, \; t > 0 \\
& u(x, 0) = \phi(x) \quad 0 < x < \pi \\
& u_x(0, t) = g(t) \quad t > 0 \\
& u(\pi, t) = h(t) \quad t > 0.
\end{align*}
\]
5. (10 points) Let $\Omega$ be an open, bounded subset of $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Assume $f(x, y) \geq 0$ for all $(x, y) \in \Omega$. Suppose $u$ is a solution of Poisson’s equation

$$\begin{cases}
  u_{xx} + u_{yy} = f(x, y) & (x, y) \in \Omega \subset \mathbb{R}^2 \\
  u(x, y) = g(x, y) & (x, y) \in \partial \Omega.
\end{cases}$$

Show that

$$\max_{\Omega} u(x, y) = \max_{\partial \Omega} g(x, y).$$
6. (12 points) Determine whether the following statements are true or false. Provide a reason for your answer.

(a) Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Let $\Omega^c \equiv \mathbb{R}^n \setminus \Omega$. There exists at most one solution of

$$\begin{cases} 
\Delta u = 0 & x \in \Omega^c \\
 u = g & x \in \partial \Omega 
\end{cases}$$

(b) Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Suppose $u$ is a solution of

$$\begin{cases} 
u_t - \Delta u = 0 & x \in \Omega, \ t > 0 \\
u(x,0) = \phi(x) \\
u(x,t) = 0 & x \in \partial \Omega. 
\end{cases}$$

Then

$$\int_{\Omega} u^2(x,t) \, dx = \int_{\Omega} \phi^2(x) \, dx.$$
(c) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a bounded function. There exits at most one bounded solution $u$ of

$$
\begin{cases}
  u_t - \Delta u = 0 & x \in \mathbb{R}^n, \ t > 0 \\
  u(x, 0) = \phi(x).
\end{cases}
$$

(d) If $u$ is a harmonic function on the rectangle

$$
\Omega \equiv \{(x, y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\},
$$

then

$$
\int_0^a u_y(x, 0) \, dx + \int_0^b u_x(a, y) \, dy + \int_0^a u_y(x, b) \, dx + \int_0^b u_x(0, y) \, dy = 0.
$$
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