Math 220B **Midterm Exam Solutions** **Summer 2002**

1. (8 points) Let $\phi \in L^2(\mathbb{R})$. Use the Fourier transform to solve

$$\begin{cases} u_t - (\sin t)u_x = 0 & -\infty < x < \infty,\ t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

Answer: Taking the Fourier transform with respect to the spatial variable, we have

$$\begin{aligned} &\widehat{u_t} = \sin(t)\widehat{u_x} \\ &\implies \widehat{u}_t = i\xi \sin(t)\widehat{u} \\ &\implies \frac{\partial \widehat{u}}{\widehat{u}} = i\xi \sin(t)\, dt \\ &\implies \ln \widehat{u} = -i\xi \cos(t) + C \\ &\implies \widehat{u}(\xi, t) = Ce^{-i\xi \cos(t)}. \end{aligned}$$

The initial condition

$$u(x,0) = \phi(x) \implies \widehat{u}(\xi, 0) = \widehat{\phi}(\xi).$$

Therefore, we have

$$\begin{aligned} &\widehat{u}(\xi, 0) = Ce^{-i\xi} = \widehat{\phi}(\xi) \\ &\implies C = e^{i\xi}\widehat{\phi}(\xi) \\ &\implies \widehat{u}(\xi, t) = \widehat{\phi}(\xi)e^{i\xi - i\xi\cos(t)}. \end{aligned}$$

Now using the fact that $u(x,t) = \check{\widehat{u}}(\xi, t)$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\widehat{u}(\xi,t)\, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\widehat{\phi}(\xi)e^{i\xi - i\xi\cos(t)}\, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi(x+1-\cos(t))}\widehat{\phi}(\xi)\, d\xi \end{aligned}$$

But, using the fact that $\phi(y) = \check{\widehat{\phi}}(\xi)$, we know that

$$\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iy\xi}\widehat{\phi}(\xi)\, d\xi.$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi(x+1-\cos(t))}\widehat{\phi}(\xi)\, d\xi = \phi(x + 1 - \cos(t)).$$

Therefore, our solution is given by

$$\boxed{u(x,t) = \phi(x + 1 - \cos(t)).}$$

2. (12 points) Let Ω be the upper half of the unit disk in $\mathbb{R}^2$. That is, let

$$\Omega \equiv \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y > 0\}.$$

Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x, y) \in \Omega \\ u(r, 0) = 0 = u(r, \pi) \\ u(1, \theta) = \theta(\theta - \pi). \end{cases}$$

Answer: First, we write the equation in polar coordinates,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Now, using separation of variables, we look for a solution of the form $u(r, \theta) = R(r)\Theta(\theta)$. Plugging this into our equation, we have

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Dividing by $R\Theta$ and multiplying by r^2, we get

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Our boundary condition

$$u(0, \theta) = 0 = u(r, \theta)$$

leads us to the eigenvalue problem

$$\begin{cases} -\Theta'' = \lambda\Theta & 0 < \theta < \pi \\ \Theta(0) = 0 = \Theta(\pi). \end{cases}$$

We know the solutions of this eigenvalue problem are

$$\Theta_n(\theta) = \sin(n\theta) \qquad \lambda_n = n^2, n = 1, 2, \ldots$$

Now we look at our equation for R_n, for $n = 1, 2, \ldots$. We know

$$r^2 R_n'' + rR_n' - n^2 R_n = 0$$

has solutions

$$R_n = C_n r^n + D_n r^{-n} \qquad n = 1, 2, \ldots$$

As we do not want our solution to blow up as $r \to 0$, we discard the solutions r^{-n}. Therefore, our solutions for R_n are

$$R_n(r) = C_n r^n \qquad n = 1, 2, \ldots$$

Therefore, we let

$$\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty} R_n(r)\Theta_n(\theta) \\ &= \sum_{n=1}^{\infty} C_n r^n \sin(n\theta). \end{aligned}$$

Our other boundary condition $u(1,\theta) = \theta(\theta - \pi)$ implies we want to find constants C_n such that

$$u(1,\theta) = \sum_{n=0}^{\infty} C_n \sin(n\theta) = \theta(\theta - \pi).$$

This is the Fourier sine series for our boundary data. We know our coefficients C_n must be given by

$$\begin{aligned} C_n &= \frac{\langle \theta(\theta-\pi), \sin(n\theta)\rangle}{\langle \sin(n\theta), \sin(n\theta)\rangle} \\ &= \frac{\int_0^\pi \theta(\theta-\pi)\sin(n\theta)\,d\theta}{\int_0^\pi \sin^2(n\theta)\,d\theta}. \end{aligned}$$

Now we need to evaluate these integrals. First, we know that

$$\int_0^\pi \sin^2(n\theta)\,d\theta = \frac{\pi}{2}.$$

Next, we look at

$$\int_0^\pi \theta(\theta-\pi)\sin(n\theta)\,d\theta = \int_0^\pi \theta^2 \sin(n\theta)\,d\theta - \pi \int_0^\pi \theta \sin(n\theta)\,d\theta \equiv I + J.$$

First, we look at term I. Integrating by parts, we have

$$\begin{aligned} \int_0^\pi \theta^2 \sin(n\theta)\,d\theta &= -\left.\frac{\theta^2 \cos(n\theta)}{n}\right|_0^\pi + \frac{2}{n}\int_0^\pi \theta\cos(n\theta)\,d\theta \\ &= \frac{-\pi^2\cos(n\pi)}{n} + \frac{2}{n}\left[\left.\frac{\theta\sin(n\theta)}{n}\right|_0^\pi - \int_0^\pi \frac{\sin(n\theta)}{n}\,d\theta\right] \\ &= \frac{-\pi^2(-1)^n}{n} + \frac{2}{n}\left[\left.\frac{\cos(n\theta)}{n^2}\right|_0^\pi\right] \\ &= \frac{-\pi^2(-1)^n}{n} + \frac{2}{n}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \end{aligned}$$

Therefore,

$$\int_0^\pi \theta^2 \sin(n\theta)\,d\theta = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & n \text{ odd} \\ \dfrac{-\pi^2}{n} & n \text{ even.} \end{cases}$$

Next, we look at term J. Integrating by parts, we have

$$-\pi \int_0^\pi \theta \sin(n\theta)\,d\theta = -\pi \left[\frac{-\theta\cos(n\theta)}{n}\Big|_0^\pi - \int_0^\pi \frac{\cos(n\theta)}{n}\,d\theta \right]$$
$$= -\pi \left[\frac{-\pi(-1)^n}{n} + \frac{\sin(n\theta)}{n^2}\Big|_0^\pi \right].$$

Therefore,

$$-\pi \int_0^\pi \theta \sin(n\theta)\,d\theta = \begin{cases} -\dfrac{\pi^2}{n} & n \text{ odd} \\ \dfrac{\pi^2}{n} & n \text{ even.} \end{cases}$$

Adding I and J, we have

$$\int_0^\pi \theta(\theta - \pi)\,d\theta = \begin{cases} -\dfrac{4}{n^3} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Therefore, we conclude that

$$C_n = \begin{cases} -\dfrac{8}{\pi n^3} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

And, therefore, our solution is given by

$$\boxed{u(r,\theta) = -\sum_{n\,\text{odd}} \frac{8}{\pi n^3} r^n \sin(n\theta).}$$

3. (6 points) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $\alpha > 0$. Prove uniqueness of solutions to the following problem,

$$\begin{cases} \Delta u - \alpha u = f & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

Answer: Suppose there are two solutions u and v. Let $w = u - v$. Then w is a solution of

$$\begin{cases} \Delta w - \alpha w = 0 & x \in \Omega \\ \frac{\partial w}{\partial \nu} = 0 & x \in \partial\Omega \end{cases}$$

Multiplying this equation by w and integrating over Ω, we have

$$0 = \int_\Omega w(\Delta w - \alpha w)\,dx$$
$$= -\int_\Omega |\nabla w|^2\,dx - \int_\Omega \alpha w^2\,dx + \int_\Omega w\frac{\partial w}{\partial \nu}\,dS(x)$$
$$= -\int_\Omega |\nabla w|^2\,dx - \int_\Omega \alpha w^2\,dx.$$

Therefore, $|\nabla w| = 0$ and $w = 0$ in Ω, which implies $u \equiv v$.

4. (12 points) Solve the following initial/boundary value problem,

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u(x,0) = \phi(x) & 0 < x < \pi \\ u_x(0,t) = g(t) & t > 0 \\ u(\pi,t) = h(t) & t > 0. \end{cases}$$

Answer: First, we introduce a function $\mathcal{U}(x,t)$ such that

$$\mathcal{U}_x(0,t) = g(t)$$
$$\mathcal{U}(\pi,t) = h(t).$$

In particular, let

$$\mathcal{U}(x,t) = g(t)(x-\pi) + h(t).$$

Now, suppose $u(x,t)$ is a solution of the initial/boundary value problem. Then let $v(x,t) = u(x,t) - \mathcal{U}(x,t)$. Then v is a solution of

$$(*) \begin{cases} v_t - v_{xx} = f(x,t) & 0 < x < \pi,\ t > 0 \\ v(x,0) = \psi(x) & 0 < x < \pi \\ v_x(0,t) = 0 & t > 0 \\ v(\pi,t) = 0 & t > 0. \end{cases}$$

where

$$f(x,t) = -\mathcal{U}_t(x,t) = -g'(t)(x-\pi) + h'(t)$$
$$\psi(x) = \phi(x) - \mathcal{U}(x,0) = \phi(x) - (g(0)(x-\pi) + h(0)).$$

To solve this inhomogeneous problem, we start by looking for the solution operator associated with the homogeneous problem,

$$\begin{cases} w_t - w_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ w(x,0) = \psi(x) & 0 < x < \pi \\ w_x(0,t) = 0 & t > 0 \\ w(\pi,t) = 0 & t > 0. \end{cases}$$

Using separation of variables, we look for a solution of the form $w(x,t) = X(x)T(t)$. Plugging this into the equation, we have

$$XT' - X''T = 0.$$

Dividing by XT, we have

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

We start by solving the eigenvalue problem,

$$\begin{cases} X'' = -\lambda X & 0 < x < \pi \\ X'(0) = 0 = X(\pi). \end{cases}$$

Looking for positive eigenvalues $\lambda = \beta^2 > 0$, we see that

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

The boundary condition

$$X'(0) = 0 \implies B = 0.$$

The boundary condition

$$X(\pi) = 0 \implies A\cos(\beta\pi) = 0 \implies \beta_n = \left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2, \ldots.$$

Therefore, our eigenfunctions are

$$X_n(x) = \cos\left(\left(n + \frac{1}{2}\right)x\right) \qquad \lambda_n = \left(n + \frac{1}{2}\right)^2.$$

By a quick check, we see that there are no negative eigenvalues.

Now, the solution for our equation for T_n is given by

$$T_n(t) = C_n e^{-\lambda_n t}.$$

Therefore, we let

$$w(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = \sum_{n=0}^{\infty} C_n \cos\left(\left(n + \frac{1}{2}\right)x\right) e^{-(n+\frac{1}{2})^2 t}.$$

The initial condition $w(x,0) = \psi(x)$ implies the coefficients C_n are given by

$$C_n = \frac{\langle \cos((n+\frac{1}{2})x), \psi(x)\rangle}{\langle \cos((n+\frac{1}{2})x), \cos((n+\frac{1}{2})x)\rangle} = \frac{\int_0^\pi \cos((n+\frac{1}{2})x)\psi(x)\,dx}{\int_0^\pi \cos^2((n+\frac{1}{2})x)\,dx}.$$

Therefore, the solution operator associated with the homogeneous problem is given by

$$S(t)\psi = \sum_{n=0}^{\infty} C_n \cos\left(\left(n + \frac{1}{2}\right)x\right) e^{-(n+\frac{1}{2})^2 t}$$

where

$$C_n = \frac{\int_0^\pi \cos((n+\frac{1}{2})x)\psi(x)\,dx}{\int_0^\pi \cos^2((n+\frac{1}{2})x)\,dx}.$$

By Duhamel's principle, the solution for the nonhomogeneous problem (*) is given by

$$v(x,t) = S(t)\psi + \int_0^t S(t-s)f(x,s)\,ds.$$

Therefore,

$$\boxed{\begin{aligned} v(x,t) = &\sum_{n=0}^{\infty} C_n \cos\left(\left(n + \frac{1}{2}\right)x\right) e^{-(n+\frac{1}{2})^2 t} \\ &+ \int_0^t \sum_{n=0}^{\infty} D_n(s) \cos\left(\left(n + \frac{1}{2}\right)x\right) e^{-(n+\frac{1}{2})^2 (t-s)} \end{aligned}}$$

where

$$\boxed{C_n = \frac{\int_0^\pi \cos((n+\frac{1}{2})x)\psi(x)\,dx}{\int_0^\pi \cos^2((n+\frac{1}{2})x)\,dx}}$$

and

$$\boxed{D_n(s) = \frac{\int_0^\pi \cos((n+\frac{1}{2})x)f(x,s)\,dx}{\int_0^\pi \cos^2((n+\frac{1}{2})x)\,dx}}$$

for ψ and f defined above. Finally, using the definition of v, we conclude that

$$\boxed{u(x,t) = v(x,t) + \mathcal{U}(x,t)}$$

with v and $\mathcal{U}$ as defined above.

5. (10 points) Let Ω be an open, bounded subset of $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Assume $f(x,y) \geq 0$ for all $(x,y) \in \Omega$. Suppose u is a solution of Poisson's equation

$$\begin{cases} u_{xx} + u_{yy} = f(x,y) & (x,y) \in \Omega \subset \mathbb{R}^2 \\ u(x,y) = g(x,y) & (x,y) \in \partial\Omega. \end{cases}$$

Show that

$$\max_{\overline{\Omega}} u(x,y) = \max_{\partial\Omega} g(x,y).$$

Answer: Let

$$M \equiv \max_{\partial\Omega} g(x,y).$$

Fix $\epsilon > 0$. Introduce a new function

$$v(x,y) = u(x,y) + \epsilon(x^2+y^2).$$

We claim that

$$\max_{\overline{\Omega}} v(x,y) \leq M + \epsilon \max_{\overline{\Omega}}(x^2+y^2).$$

Assuming, this claim for now, we conclude that for all $(x,y) \in \Omega$,

$$\begin{aligned} u(x,y) &= v(x,y) - \epsilon(x^2+y^2) \\ &\leq M + \epsilon(\max_{\overline{\Omega}}(x^2+y^2) - (x^2+y^2)). \end{aligned}$$

Since this is true for all ϵ, we conclude that

$$u(x,y) \leq M$$

for all $(x,y) \in \overline{\Omega}$. Then the result follows.

Therefore, we just need to show that

$$(*) \qquad v(x,y) \leq M + \max_{\overline{\Omega}}(x^2+y^2)$$

for all $(x, y) \in \overline{\Omega}$. Clearly, (*) is true for all $(x, y) \in \partial\Omega$. Therefore, we just need to check $(x, y) \in \Omega$. By definition of v, we see that

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} + 2\epsilon \geq 2\epsilon > 0.$$

Now suppose v has an interior maximum at some interior point (x, y). But, then $v_{xx}, v_{yy} \leq 0$, which contradicts the above inequality. Therefore, v cannot have an interior maximum. And, therefore, for all (x, y) in $\overline{\Omega}$, (*) holds. Therefore, the result follows.

6. (12 points) Determine whether the following statements are true or false. **Provide a reason for your answer.**

   (a) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $\Omega^c \equiv \mathbb{R}^n \backslash \Omega$. There exists at most one solution of

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ u = g & x \in \partial\Omega \end{cases}$$

   Answer: $\boxed{False}$ Counterexample: Let $n = 1$. Let $\Omega = (0, 1)$. Let $g = 1$ for $x = 0, 1$. Let Let

$$u_1(x) = 1 \qquad x \leq 0, x \geq 1.$$

$$u_2(x) = \begin{cases} x & x \geq 1 \\ x + 1 & x \leq 0. \end{cases}$$

   Both u_1, u_2 are solutions to our problem.

   (b) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Suppose u is a solution of

$$\begin{cases} u_t - \Delta u = 0 & x \in \Omega,\ t > 0 \\ u(x, 0) = \phi(x) & \\ u(x, t) = 0 & x \in \partial\Omega. \end{cases}$$

   Then

$$\int_\Omega u^2(x, t)\, dx = \int_\Omega \phi^2(x)\, dx.$$

   Answer: $\boxed{False}$

$$\begin{aligned} \frac{d}{dt} \int_\Omega u^2(x, t)\, dx &= 2 \int_\Omega u u_t\, dx \\ &= 2 \int_\Omega u \Delta u\, dx \\ &= -2 \int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u \frac{\partial u}{\partial \nu}\, dS(x) \\ &= -2 \int_\Omega |\nabla u|^2\, dx. \end{aligned}$$

   Therefore,

$$\frac{d}{dt} \int_\Omega u^2(x, 0)\, dx = -2 \int_\Omega |\nabla \phi|^2\, dx < 0$$

   unless ϕ is constant.

(c) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a bounded function. There exits at most one bounded solution u of

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

Answer: $\boxed{True}$ This follows from the uniqueness theorem for solutions of the heat equation on $\mathbb{R}^n$. In particular, we know there exits at most one solution $u(x,t)$ which satisfies the growth estimate

$$|u(x,t)| \leq Ae^{a|x|^2}$$

for some constants A, a. Therefore, in particular, there exits at most one bounded solution.

(d) If u is a harmonic function on the rectangle

$$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\},$$

then

$$\int_0^a u_y(x,0)\,dx + \int_0^b u_x(a,y)\,dy + \int_0^a u_y(x,b)\,dx + \int_0^b u_x(0,y)\,dy = 0.$$

Answer: $\boxed{False}$ Let $u(x,y) = x$. This function is harmonic on Ω. Clearly, $u_y = 0$. But, $u_x(0,y) = 1 = u_x(a,y)$. Therefore,

$$\int_0^a u_y(x,0)\,dx + \int_0^b u_x(a,y)\,dy + \int_0^a u_y(x,b)\,dx + \int_0^b u_x(0,y)\,dy = 2b \neq 0.$$

Scratch Paper