Math 220B **Midterm Exam Solutions** **Summer 2003**

1. (14 points)

   (a) Find all eigenvalues and eigenfunctions of

$$\begin{cases} X'' = -\lambda X & 0 < x < 1 \\ X'(0) - X(0) = 0 \\ X(1) = 0. \end{cases}$$

Justify your answer.

Answer: First, we look for positive eigenvalues $\lambda = \beta^2 > 0$. In this case,

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

The boundary condition

$$X'(0) - X(0) = 0 \implies B\beta - A = 0.$$

The boundary condition

$$X(1) = 0 \implies A\cos(\beta) + B\sin(\beta) = 0.$$

Combining these two equations, we have

$$\tan(\beta) = -\beta.$$

Therefore, our positive eigenvalues are given by

$$\boxed{\lambda_n = \beta_n^2 \text{ where } \tan(\beta_n) = -\beta_n.}$$

The corresponding eigenfunctions are given by

$$\boxed{X_n(x) = \beta_n\cos(\beta_n x) + \sin(\beta_n x).}$$

If $\lambda = 0$, then

$$X(x) = Ax + B.$$

The boundary condition

$$X'(0) - X(0) = 0 \implies A = B.$$

The boundary condition

$$X(1) = 0 \implies A = -B.$$

Combining these two equations, we see that $A = 0 = B$. Therefore, $\lambda = 0$ is not an eigenvalue.

Next, we check if we have any negative eigenvalues. Using the result from a previous homework, we consider

$$X'X\big|_{x=0}^{x=1}.$$

We recall that if this quantity is non-positive, then we have no negative eigenvalues. We see that

$$\begin{aligned} X'X\big|_{x=0}^{x=1} &= X'(1)X(1) - X'(0)X(0) \\ &= 0 - X^2(0) \le 0. \end{aligned}$$

Therefore, we have no negative eigenvalues.

(b) Solve the following initial/boundary value problem

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < 1, t > 0 \\ u(x,0) = 0 \\ u_x(0,t) - u(0,t) = 0 \\ u(1,t) = t. \end{cases}$$

You do not need to explicitly evaluate any integrals.

Answer: First, we will shift the boundary data. Introduce a function $\mathcal{U}(x,t)$ such that $\mathcal{U}_x(0,t) - \mathcal{U}(0,t) = 0$ and $\mathcal{U}(1,t) = t$. We see that the function

$$\boxed{\mathcal{U}(x,t) = \frac{1}{2}t(x+1)}$$

satisfies these conditions. Assuming u is a solution of the problem above, let $v(x,t) = u(x,t) - \mathcal{U}(x,t)$. Then v satisfies

$$\begin{cases} v_t - v_{xx} = -\frac{1}{2}(x+1) & 0 < x < 1 \\ v(x,0) = 0 \\ v_x(0,t) - v(0,t) = 0 \\ v(1,t) = 0. \end{cases}$$

In order to solve this inhomogeneous problem, we will first derive the solution operator for the homogeneous problem and then apply Duhamel's principle.

Using separation of variables, we look for a solution of the homogeneous problem the form $u(x,t) = X(x)T(t)$. Plugging this function into our PDE, we have

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

In particular, we have the eigenvalue problem

$$\begin{cases} X'' = -\lambda X & 0 < x < 1 \\ X'(0) - X(0) = 0 \\ X(1) = 0. \end{cases}$$

The eigenfunctions $X_n(x)$ and eigenvalues λ_n are given in part (a).

The solutions for our equation for T_n are given by $T_n(t) = C_n e^{-\lambda_n t}$.

Therefore, the solution operator for the homogeneous problem is given by

$$S(t)\phi = \sum_{n=1}^{\infty} C_n X_n(x) e^{-\lambda_n t}$$

where

$$C_n = \frac{\langle X_n(x), \phi \rangle}{\langle X_n(x), X_n(x) \rangle}.$$

Therefore, by Duhamel's principle the solution of the inhomogeneous problem is given by

$$\boxed{v(x,t) = \int_0^t \sum_{n=1}^{\infty} D_n(s) X_n(x) e^{-\lambda_n(t-s)}\, ds}$$

where

$$\boxed{D_n(s) = \frac{\langle X_n(x), f(x,s)\rangle}{\langle X_n(x), X_n(x)\rangle}}$$

for $\boxed{f(x,t) = -\frac{1}{2}(x+1)}$. Finally, we conclude that

$$\boxed{u(x,t) = v(x,t) + \mathcal{U}(x,t)}$$

for v and $\mathcal{U}$ defined above.

2. (10 points) Find

$$\lim_{n\to+\infty} \sqrt{n} e^{-nx^2}$$

in the sense of distributions. Prove your answer.

Answer: First, we note that

$$\sqrt{n} e^{-nx^2} \to \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0. \end{cases}$$

Next, we note that

$$\int_{-\infty}^{\infty} \sqrt{n} e^{-nx^2}\, dx = \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}.$$

Using the above motivation, we will show that $\sqrt{n} e^{-nx^2} \to \sqrt{\pi}\delta_0$ in the sense of distributions.

Define the distribution $F_n : \mathcal{D} \to \mathbb{R}$ such that

$$(F_n, \phi) \equiv \int_{-\infty}^{\infty} \sqrt{n} e^{-nx^2} \phi(x)\, dx.$$

To show that $F_n \rightharpoonup \sqrt{\pi}\delta_0$, we need to show that

$$(F_n, \phi) \to \sqrt{\pi}(\delta_0, \phi) = \sqrt{\pi}\phi(0)$$

as $n \to +\infty$ for all $\phi \in \mathcal{D}$. We proceed as follows.

$$\begin{aligned} \left|(F_n, \phi) - \sqrt{\pi}\phi(0)\right| &= \left| \int_{-\infty}^{\infty} \sqrt{n} e^{-nx^2} \phi(x)\, dx - \sqrt{\pi}\phi(0) \right| \\ &= \left| \int_{-\infty}^{\infty} \sqrt{n} e^{-nx^2} \phi(x)\, dx - \int_{-\infty}^{\infty} \sqrt{n} e^{-nx^2} \phi(0)\, dx \right| \\ &= \left| \int_{-\infty}^{\infty} \sqrt{n} e^{-nx^2} [\phi(x) - \phi(0)]\, dx \right|. \end{aligned}$$

First, we note that ϕ has compact support. Therefore, let K denote the support of ϕ. Now we break this integral into two pieces: $B(0,\delta)$ and $K\backslash B(0,\delta)$. First, on $B(0,\delta)$, we see that

$$\begin{aligned}\left|\int_{-\delta}^{\delta} \sqrt{n}e^{-nx^2}[\phi(x)-\phi(0)]\,dx\right| &\leq |\phi(x)-\phi(0)|_{L^\infty(-\delta,\delta)} \int_{-\infty}^{\infty} \sqrt{n}e^{-nx^2}\,dx \\ &< \frac{\epsilon}{2},\end{aligned}$$

for any $\epsilon > 0$ by choosing δ sufficiently small. (Here, we have used the fact that ϕ is a continuous function, and that $\int \sqrt{n}e^{-nx^2}$ is bounded.

Now δ is fixed. We look at the integral over $K\backslash B(0,\delta)$. Here we have

$$\begin{aligned}\left|\int_{K\backslash B(0,\delta)} \sqrt{n}e^{-nx^2}[\phi(x)-\phi(0)]\,dx\right| &\leq C|\sqrt{n}e^{-nx^2}|_{L^\infty(K\backslash B(0,\delta))} \\ &= C\sqrt{n}e^{-n\delta^2} \\ &< \frac{\epsilon}{2}\end{aligned}$$

by choosing n sufficiently large, using the fact that $\sqrt{n}e^{-n\delta^2} \to 0$ as $n \to +\infty$.

3. (10 points) Use the Fourier transform to solve

$$\begin{cases} u_t - e^{-t}u_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

You may use the fact that

$$f(x) = e^{-\epsilon x^2} \implies \widehat{f}(\xi) = \frac{1}{\sqrt{2\epsilon}}e^{-\xi^2/4\epsilon}.$$

Simplify your answer as much as possible.

Answer:

$$\begin{aligned}\widehat{u}_t - \widehat{e^{-t}u_{xx}} &= 0 \\ \implies \widehat{u}_t - e^{-t}(i\xi)^2\widehat{u} &= 0 \\ \implies \widehat{u}_t + \xi^2 e^{-t}\widehat{u} &= 0.\end{aligned}$$

Solving this ODE, we have

$$\ln \widehat{u} = \xi^2 e^{-t} + C.$$

Therefore,

$$\widehat{u}(\xi,t) = Ce^{\xi^2 e^{-t}}.$$

The initial condition $u(x,0) = \phi(x)$ implies $\widehat{u}(\xi,0) = \widehat{\phi}(\xi)$. Therefore,

$$\widehat{u}(\xi,t) = Ce^{\xi^2} = \widehat{\phi}(\xi) \implies C = \widehat{\phi}e^{-\xi^2}.$$

Now

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{\phi}(\xi) e^{-\xi^2} e^{\xi^2 e^{-t}} \, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{\phi}(\xi) e^{\xi^2 [e^{-t}-1]} \, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi} \phi(y) \, dy e^{\xi^2 [e^{-t}-1]} \, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(y-x)\xi} e^{-\xi^2[1-e^{-t}]} \, d\xi \right) \phi(y) \, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(y-x) \phi(y) \, dy
\end{aligned}
$$

where $f(\xi) = e^{-\xi^2[1-e^{-t}]}$. From the fact stated above,

$$
f(\xi) = e^{-\xi^2[1-e^{-t}]} \implies \widehat{f}(y-x) = \frac{1}{(2(1-e^{-t}))^{1/2}} e^{-|y-x|^2/4(1-e^{-t})}.
$$

Therefore, our solution is given by

$$
\boxed{u(x,t) = \frac{1}{\sqrt{4\pi(1-e^{-t})}} \int_{-\infty}^{\infty} e^{-|x-y|^2/4(1-e^{-t})} \phi(y) \, dy.}
$$

4. (10 points)

   (a) Let Ω be an open, bounded subset of $\mathbb{R}^n$. Consider the following boundary-value problem,

$$
\begin{cases} \Delta u + \sum_{i=1}^n b_i u_{x_i} + cu = f & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases} \tag{1}
$$

   Prove there exists at most one solution of (2) in the case that $c \leq 0$.

   Answer: Suppose there are two solutions u and v. Let $w = u - v$. Then w satisfies

$$
\begin{cases} \Delta w + \sum_{i=1}^n b_i w_{x_i} + cw = 0 & x \in \Omega \\ w = 0 & x \in \partial\Omega. \end{cases} \tag{2}
$$

   Now multiplying our PDE by w and integrating over Ω, we have

$$
\begin{aligned}
0 &= \int_\Omega w(\Delta w + \sum_{i=1}^n b_i w_{x_i} + cw) \, dx \\
&= -\int_\Omega |\nabla w|^2 \, dx + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS + \frac{1}{2} \sum_{i=1}^n b_i \int_\Omega (w^2)_{x_i} \, dx + c \int_\Omega w^2 \, dx \\
&= -\int_\Omega |\nabla w|^2 \, dx + \frac{1}{2} \sum_{i=1}^n b_i \int_{\partial\Omega} w^2 \nu_i \, dS + c \int_\Omega w^2 \, dx \\
&= -\int_\Omega |\nabla w|^2 \, dx + c \int_\Omega w^2 \, dx.
\end{aligned}
$$

Since $c \leq 0$, we know that the only way the above equality can hold is if each term is identically zero. Therefore,

$$\int_\Omega |\nabla w|^2 \, dx = 0 = \int_\Omega w^2 \, dx,$$

which implies $w \equiv 0$. Therefore, $u = v$.

(b) Give a counterexample to show solutions of (2) need not be unique if $c > 0$. In particular, find a set Ω for which there are two solutions of

$$\begin{cases} \Delta u + u = 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Answer: Let $\Omega = (0, \pi)$. Then consider

$$\begin{cases} u'' + u = 0 & x \in (0, \pi) \\ u(0) = 0 = u(\pi). \end{cases}$$

We know that $u(x) = C\sin(x)$ is a solution of this boundary-value problem for any C. Therefore, the solution is not unique.

5. (12 points) Consider the eigenvalue problem

$$\begin{cases} -(p(x)u')' = \lambda r(x) u & x \in (a, b) \\ u(a) = 0 = u(b) \end{cases}$$

where $p(x) > 0$ and $r(x) > 0$.

(a) Prove orthogonality of eigenfunctions (corresponding to distinct eigenvalues) with respect to the weight function $r(x)$. That is, show that if X_n and X_m are eigenfunctions corresponding to the eigenvalues $\lambda_n \neq \lambda_m$, then

$$\int_a^b X_n(x) X_m(x) r(x) \, dx = 0.$$

Answer:

$$\begin{aligned} \lambda_n \int_a^b X_n X_m r(x) \, dx &= -\int_a^b (p(x) X_n')' X_m \, dx \\ &= \int_a^b p(x) X_n' X_m' \, dx - p X_n' X_m \Big|_{x=a}^{x=b} \\ &= \int_a^b X_n' p X_m' \, dx \\ &= -\int_a^b X_n (p X_m')' \, dx + X_n p(x) X_m' \Big|_{x=a}^{x=b} \\ &= -\int_a^b X_n \lambda_m r(x) X_m \, dx \\ &= \lambda_m \int_a^b X_n X_m r(x) \, dx. \end{aligned}$$

Therefore,

$$(\lambda_n - \lambda_m) \int_a^b X_n X_m r(x)\, dx = 0.$$

Since $\lambda_n \neq \lambda_m$, we must have

$$\int_a^b X_m X_m r(x)\, dx = 0,$$

as claimed.

(b) Consider the variable-coefficient heat equation

$$\begin{cases} r(x)u_t - (p(x)u_x)_x = 0 & x \in (a,b) \\ u(x,0) = \phi(x) \\ u(a,t) = 0 = u(b,t) \end{cases} \tag{3}$$

where $p(x) > 0$ and $r(x) > 0$. Suppose λ_n and X_n are the eigenvalues and corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases} -(p(x)X')' = \lambda r(x)X & x \in (a,b) \\ X(a) = 0 = X(b). \end{cases}$$

Assume each eigenvalue λ_n has multiplicity one. Using the result of part (a), write the solution of (3) in terms of X_n, λ_n and ϕ.

Answer: Look for a solution of the form $u(x,t) = X(x)T(t)$. Plugging this into the equation, we have

$$\frac{T'}{T} = \frac{(pX')'}{rX} = -\lambda.$$

We are led to the eigenvalue problem

$$\begin{cases} (p(x)X')' = -\lambda r(x)X & a < x < b \\ X(a) = 0 = X(b). \end{cases}$$

By assumption, the solutions of this eigenvalue problem are given by λ_n, X_n. The solutions for the equation in T are given by $T_n(t) = C_n e^{-\lambda_n t}$. Therefore, we let

$$u(x,t) = \sum_{n=1}^{\infty} C_n X_n e^{-\lambda_n t}.$$

It remains to determine our coefficients. We want $u(x,0) = \phi(x)$. Therefore, we want to choose coefficients C_n such that

$$\phi(x) = \sum_{n=1}^{\infty} C_n X_n(x).$$

By the result from part (a), we know that the eigenfunctions X_n are orthogonal with respect to the weight function $r(x)$. Therefore, multiplying our equation by $X_m r(x)$ and integrating over (a,b), we conclude that

$$C_m = \frac{\int_a^b X_m(x)\phi(x)r(x)\, dx}{\int_a^b X_m^2(x)r(x)\, dx}.$$

6. (10 points) Solve

$$\begin{cases} u_t - u_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \sin(x). \end{cases}$$

Simplify your answer as much as possible.

Answer: We know the solution is given by convolution with the fundamental solution. Therefore,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \sin(y)\, dy.$$

To simplify, we substitute

$$\sin(y) = \frac{e^{iy} - e^{-iy}}{2i}.$$

First, we consider

$$\frac{1}{2i\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} e^{iy}\, dy = \frac{e^{ix}}{2i\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} e^{-i(x-y)}\, dy.$$

We complete the square in the exponent. We see that

$$\begin{aligned} -\frac{(x-y)^2}{4t} - i(x-y) &= -\frac{1}{4t}\left[(x-y)^2 + 4ti(x-y) + (2ti)^2 - (2ti)^2\right] \\ &= -\frac{1}{4t}[(x-y) + i2t]^2 - t. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{e^{ix}}{2i\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} e^{-i(x-y)}\, dy &= \frac{e^{ix}e^{-t}}{2i\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}[(x-y)+i2t]^2}\, dy \\ &= \frac{e^{ix-t}}{2i\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-y)^2}\, dy. \end{aligned}$$

Letting $z = y - x/\sqrt{4t}$, we see that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-y)^2}\, dy = \sqrt{4\pi t}.$$

Therefore, we have

$$\frac{1}{2i\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} e^{iy}\, dy = \frac{e^{ix-t}}{2i\sqrt{4\pi t}} \sqrt{4\pi t} = \frac{e^{ix-t}}{2i}.$$

Similarly, we note that

$$-\frac{1}{2i\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} e^{-iy}\, dy = \frac{e^{ix-t}}{2i\sqrt{4\pi t}} \sqrt{4\pi t} = -\frac{e^{-ix-t}}{2i}.$$

Therefore, we conclude that

$$u(x,t) = \frac{e^{ix-t}}{2i} - \frac{e^{-ix-t}}{2i},$$

or

$$\boxed{u(x,t) = e^{-t}\sin(x).}$$