Name: ____________________________________________________________

Please sign below in acknowledgment and acceptance of the Honor Code.

Signature: _________________________________________________________

This exam is closed notes, closed book. The exam is worth a total of 50 points. Each problem is worth 10 points. Please show all work and clearly mark your answer.

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1. Use the Fourier transform to solve
\[
\begin{cases}
    u_t = ku_{xx} - u & x \in \mathbb{R}, t > 0 \\
    u(x, 0) = \phi(x).
\end{cases}
\]
Express your answer in the form
\[
u(x, t) = \int_{-\infty}^{\infty} K(x, y, t) \phi(y) \, dy
\]
for some kernel \(K(x, y, t)\). You may use the fact that for \(x \in \mathbb{R},\)
\[
f(x) = e^{-ax^2} \implies \hat{f}(\xi) = \frac{1}{\sqrt{2a}} e^{-\xi^2/4a}.
\]
2. Show that \( u(x) = e^{-|x|} \) solves
\[-u_{xx} + u = 2\delta_0\]
in the sense of distributions.
3. (a) Let \( \Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi \} \). Solve the following boundary-value problem,

\[
\begin{align*}
\Delta u &= 0 \quad (x, y) \in \Omega \\
u(0, y) &= u(x, y) = 0 = u(\pi, y) \quad 0 < y < \pi \\
u(x, 0) &= 0 \quad 0 < x < \pi \\
u(x, \pi) &= 2\sin(x) - \sin(2x) \quad 0 < x < \pi.
\end{align*}
\]

(b) Find \( \max_{\Omega} u(x, y) \).
4. (a) Solve
\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u &= f(x) \quad 0 < x < l \\
\frac{\partial u}{\partial x}(0, t) &= 0 = \frac{\partial u}{\partial x}(l, t) \\
\frac{\partial u(x, 0)}{\partial x} &= g(x) \\
u(x, 0) &= 0 \quad 0 < x < l.
\end{align*}
\]

(b) Prove uniqueness of this solution.
5. (a) Answer true or false to the following statements. No explanation is necessary. (1 point each)

i. Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Assume $g$ satisfies the compatibility condition

$$\int_{\partial \Omega} g(y) dS(y) = 0.$$ 

There exists at most one smooth solution of

$$\begin{cases} 
\Delta u = 0 & x \in \Omega \\
\frac{\partial u}{\partial \nu} = g & x \in \partial \Omega.
\end{cases}$$

ii. If $u$ is a bounded, harmonic function on $\mathbb{R}^n$, then $u$ must be constant.

iii. There exists at most one smooth, bounded solution of

$$\begin{cases} 
u t - k\Delta u = f & x \in \mathbb{R}^n, t > 0 \\
u(x, 0) = \phi(x)
\end{cases}$$

iv. If $u$ is a smooth, harmonic function in $\Omega$ and $u = g \geq 0$ on $\partial \Omega$, then $u > 0$ in $\Omega$. 
(b) Answer the following short answer questions. (3 points each)

i. What is the derivative of the delta function?

ii. State the strong maximum principle for the heat equation.