1. Use the Fourier transform to solve
\[
\begin{align*}
\begin{cases}
    u_t &= ku_{xx} - u & x \in \mathbb{R}, t > 0 \\
    u(x, 0) &= \phi(x).
\end{cases}
\end{align*}
\]
You may use the fact that for \( x \in \mathbb{R}, \)
\[ f(x) = e^{-\epsilon |x|^2} \implies \hat{f}(\xi) = \frac{1}{\sqrt{2\epsilon}} e^{-|\xi|^2/4\epsilon}. \]

**Answer:**
\[
\begin{align*}
    u_t - ku_{xx} + u &= 0 \\
    \implies \hat{u}_t - k\hat{u}_{xx} + \hat{u} &= 0 \\
    \implies \hat{u}_t - k(i\xi)^2\hat{u} + \hat{u} &= 0 \\
    \implies \hat{u}_t + k(\xi^2 + 1)\hat{u} &= 0 \\
    \implies \frac{\hat{u}_t}{\hat{u}} &= -(k\xi^2 + 1) \\
    \implies \ln \hat{u} &= -(k\xi^2 + 1)t + C \\
    \implies \hat{u} &= Ce^{-(k\xi^2 + 1)t}.
\end{align*}
\]
The initial condition implies \( \hat{u}(\xi) = \hat{\phi}e^{-(k\xi^2 + 1)t}. \)
Using the inverse Fourier transform, we have
\[
\begin{align*}
    u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \hat{u}(\xi, t) \, d\xi \\
    &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \hat{\phi}(\xi) e^{-(k\xi^2 + 1)t} \, d\xi \\
    &= \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}(\xi) e^{-k\xi^2t} e^{ix\xi} \, d\xi \\
    &= \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi} \phi(y) \, dy \right] e^{-k\xi^2t} e^{ix\xi} \, d\xi \\
    &= \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(y-x)\xi} e^{-k\xi^2t} \, d\xi \right] \phi(y) \, dy.
\end{align*}
\]
Now
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(y-x)\xi} e^{-k\xi^2t} \, d\xi = \hat{f}(y - x)
\]
where
\[ f(\xi) = e^{-\xi^2t}. \]
Now using the fact above, we see that
\[ f(\xi) = e^{-k\xi^2t} \implies \hat{f}(x) = \frac{1}{\sqrt{2kt}} e^{-x^2/4kt}. \]
Therefore,
\[ u(x, t) = e^{-t} \sqrt{\frac{4k\pi t}{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) \, dy. \]

2. Show that \( u(x) = e^{-|x|} \) solves
\[-u_{xx} + u = 2\delta_0 \]
in the sense of distributions.

**Answer:** To say that \(-u_{xx} + u = 2\delta_0 \) in the sense of distributions, means the following. Let \( F_u \) be the distribution associated with \( u \). That is, \( F_u : \mathcal{D} \to \mathbb{R} \) such that
\[ (F_u, \phi) = \int_{-\infty}^{\infty} e^{-|x|} \phi(x) \, dx \]
for all \( \phi \in \mathcal{D} \). We need to show that
\[ ([-\partial_x^2 + I]F_u, \phi) = 2(\delta_0, \phi) = 2\phi(0). \]

Recall that the derivative of a distribution \( F \) is defined to be the distribution \( G \) such that
\[ (G, \phi) \equiv -(F, \phi') \quad \forall \phi \in \mathcal{D}. \]

Therefore,
\[ ([-\partial_x^2 + I]F_u, \phi) \equiv (F_u, [-\partial_x^2 + I]\phi). \]

Therefore, it remains to show that
\[ (F_u, [-\partial_x^2 + I]\phi) = 2\phi(0) \quad \forall \phi \in \mathcal{D}. \]

We proceed as follows.
\[ (F_u, [-\partial_x^2 + I]\phi) = \int_{-\infty}^{\infty} e^{-|x|}[-\phi''(x) + \phi(x)] \, dx \]
\[ = \int_{0}^{\infty} e^{-x}[-\phi''(x)] \, dx + \int_{-\infty}^{0} e^{x}[-\phi''(x)] \, dx + \int_{-\infty}^{\infty} e^{-|x|}\phi(x) \, dx \]
\[ \equiv A_1 + A_2 + A_3. \]

Now integrating \( A_1 \) by parts, we see
\[ \int_{0}^{\infty} e^{-x}[-\phi''(x)] \, dx = -\int_{0}^{\infty} e^{-x}\phi'(x) \, dx - e^{-x}\phi'(x)|_{0}^{\infty} \]
\[ = -\int_{0}^{\infty} e^{-x}\phi(x) \, dx - e^{-x}\phi(x)|_{0}^{\infty} - e^{-x}\phi'(x)|_{0}^{\infty} \]
\[ = -\int_{0}^{\infty} e^{-x}\phi(x) \, dx + \phi(0) + \phi'(0). \]
Similarly, integrating $A_2$ by parts, we see

\[ \int_{-\infty}^{0} e^x [-\phi''(x)] \, dx = \int_{-\infty}^{0} e^x \phi'(x) \, dx - e^x \phi'(x)|_{-\infty}^{0} \]

\[ = -\int_{-\infty}^{0} e^x \phi(x) \, dx + e^x \phi(x)|_{-\infty}^{0} - e^x \phi'(x)|_{-\infty}^{0} \]

\[ = -\int_{-\infty}^{0} e^x \phi(x) \, dx + \phi(0) - \phi'(0). \]

Now adding these terms with $A_3$, we see that

\[ (F_u, [-\partial_x^2 + I] \phi) = 2\phi(0), \]

as claimed.

3. (a) Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi \}$. Solve the following boundary-value problem,

\[ \begin{cases} 
\Delta u = 0 & (x, y) \in \Omega \\
u(0, y) = 0 = u(\pi, y) & 0 < y < \pi \\
u(x, 0) = 0 & 0 < x < \pi \\
u(x, \pi) = 2\sin(x) - \sin(2x) & 0 < x < \pi.
\end{cases} \]

**Answer:** We use separation of variables. Look for a solution of the form $u(x, y) = X(x)Y(y)$. Plugging this into the PDE, we have

\[ X''Y + XY'' = 0. \]

Dividing by $XY$, we have

\[ \frac{X''}{X} + \frac{Y''}{Y} = 0. \]

Given that we have homogeneous boundary conditions at $x = 0$ and $x = \pi$, we will write the above equation as

\[ \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda. \]

We are led to the eigenvalue problem,

\[ X'' = -\lambda X \quad 0 < x < \pi \]
\[ X(0) = 0 = X(\pi). \]

We know the eigenvalues and eigenfunctions of this problem are given by

\[ \lambda_n = n^2 \quad X_n(x) = \sin(nx) \quad \text{for } n = 1, 2, \ldots \]

Now we need to solve our equation for $Y_n$ for each $\lambda_n$. In particular, we need to solve

\[ Y_n'' - n^2Y_n = 0. \]
The solutions of this second-order ODE are given by

\[ Y_n(y) = A_n \cosh(ny) + B_n \sinh(ny). \]

The boundary condition \( u(x, 0) = 0 \) implies \( Y(0) = 0 \). Therefore,

\[ Y_n(0) = A_n = 0. \]

Therefore, for each \( n = 1, 2, \ldots \), we have found a function \( u_n(x, y) = X_n(x)Y_n(y) \) which is harmonic in \( \Omega \) and satisfies three of the boundary conditions. It remains to satisfy our fourth boundary condition. Let

\[ u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) \]

\[ = \sum_{n=1}^{\infty} B_n \sinh(ny) \sin(nx) \]

for \( B_n \) arbitrary. Now, we will choose \( B_n \) to satisfy our fourth boundary condition. We want \( u(x, \pi) = 2 \sin(x) - \sin(2x) \). That is, we want to choose \( B_n \) such that

\[ \sum_{n=1}^{\infty} B_n \sinh(n\pi) \sin(nx) = 2 \sin(x) - \sin(2x). \]

We note that the right-hand side is a finite combination of sine functions. Therefore, we just need to choose \( B_n \) such that

\[ B_1 \sinh(\pi) = 2 \]
\[ B_2 \sinh(2\pi) = -1 \]
\[ B_n = 0 \quad n = 3, 4, \ldots \]

Therefore, our solution is given by

\[ u(x, y) = \frac{2}{\sinh(\pi)} \sinh(y) \sin(x) - \frac{1}{\sinh(2\pi)} \sinh(2y) \sin(2x). \]

(b) Find \( \max_{\Omega} u(x, y) \).

**Answer:** We know from the maximum principle for harmonic functions that

\[ \max_{\Omega} u(x, y) = \max_{\partial \Omega} u(x, y). \]

Now, \( u(x, y) = 0 \) along three of the sides. We just need to check what the maximum of \( u \) is on the fourth side. On the fourth side,

\[ u(x, \pi) = 2 \sin(x) - \sin(2x). \]
To find the maximum on this side, we look for critical points of $u(x, \pi)$.

$$u_x(x, \pi) = 2 \cos(x) - 2 \cos(2x) = 0$$

$$\Rightarrow \cos(x) = \cos(2x)$$

$$\Rightarrow \cos(x) = 2 \cos^2(x) - 1$$

$$\Rightarrow (2 \cos(x) + 1)(\cos(x) - 1) = 0$$

$$\Rightarrow x = 0, 2\pi/3.$$ 

We see that $u(0, \pi) = 0$. We see that $u(2\pi/3, \pi) = 2 \sin(2\pi/3) - \sin(4\pi/3) = 2\sqrt{3}/2 + \sqrt{3}/2 = 3\sqrt{3}/2$. Therefore,

$$\max_{\Omega} u(x, y) = 3\sqrt{3}/2.$$ 

4. (a) Solve

$$\begin{cases}
  u_t - u_{xx} + u = f(x) & 0 < x < l \\
  u(x, 0) = g(x) & 0 < x < l \\
  u_x(0, t) = 0 = u_x(l, t).
\end{cases}$$

**Answer:** First, we will solve the homogeneous problem

$$u_t - u_{xx} + u = 0.$$ 

Using separation of variables, we are led to the eigenvalue problem

$$\begin{cases}
  X'' = (1 - \lambda)X & 0 < x < l \\
  X'(0) = 0 = X'(l).
\end{cases}$$

Letting $\mu = \lambda - 1$, this becomes the eigenvalue problem

$$\begin{cases}
  X'' = -\mu X & 0 < x < l \\
  X'(0) = 0 = X'(l).
\end{cases}$$

We know the eigenfunctions and eigenvalues for this problem are

$$X_n(x) = \cos \left( \frac{n\pi}{l} x \right), \quad \mu_n = \left( \frac{n\pi}{l} \right)^2, \quad n = 0, 1, 2, \ldots.$$ 

Therefore, $\lambda_n = 1 + \mu_n = 1 + \left( \frac{n\pi}{l} \right)^2$. Now we need to solve the equation for $T_n$,

$$T'_n = -\lambda_n T \implies T_n(t) = C_n e^{-(1+(n\pi/l)^2)t}.$$ 

Let

$$u(x, t) = \sum_{n=0}^\infty C_n \cos \left( \frac{n\pi}{l} x \right) e^{-(1+(n\pi/l)^2)t}.$$
In order for our initial condition to be satisfied, we need

\[ u(x, 0) = \sum_{n=0}^{\infty} C_n \cos \left( \frac{n\pi}{l} x \right) = g(x). \]

This is just the Fourier cosine series for \( \phi \). The coefficients \( C_n \) are given by

\[ C_n = \frac{\langle \cos \left( \frac{n\pi}{l} x \right), g(x) \rangle}{\langle \cos \left( \frac{n\pi}{l} x \right), \cos \left( \frac{n\pi}{l} x \right) \rangle} = \frac{\int_0^l \cos \left( \frac{n\pi}{l} x \right) g(x) \, dx}{\int_0^l \cos^2 \left( \frac{n\pi}{l} x \right) \, dx}. \]

Therefore,

\[ C_n = \begin{cases} \frac{2}{l} \int_0^l \cos \left( \frac{n\pi}{l} x \right) g(x) \, dx & n = 1, 2, \ldots \\ \frac{1}{l} \int_0^l g(x) \, dx & n = 0 \end{cases} \]

Therefore, the solution operator \( S(t) \) associated with the homogeneous equation is the operator such that

\[ S(t)\phi = \sum_{n=0}^{\infty} C_n \cos \left( \frac{n\pi}{l} x \right) e^{-\left(1+(n\pi/l)^2\right)t} \]

with \( C_n \) as defined above. Using Duhamel’s principle, we know that the solution of the inhomogeneous equation is given by

\[ u(x, t) = S(t)\phi + \int_0^t S(t-s) f(s) \, ds. \]

Therefore, the solution of our problem is given by

\[
\begin{align*}
  u(x, t) &= \sum_{n=0}^{\infty} C_n \cos \left( \frac{n\pi}{l} x \right) e^{-\left(1+(n\pi/l)^2\right)t} \\
  &\quad + \int_0^t \sum_{n=0}^{\infty} D_n(s) \cos \left( \frac{n\pi}{l} x \right) e^{-\left(1+(n\pi/l)^2\right)(t-s)} \, ds
\end{align*}
\]

where

\[ C_n = \begin{cases} \frac{2}{l} \int_0^l \cos \left( \frac{n\pi}{l} x \right) g(x) \, dx & n = 1, 2, \ldots \\ \frac{1}{l} \int_0^l \phi(x) \, dx & n = 0 \end{cases} \]

and

\[ D_n(s) = \begin{cases} \frac{2}{l} \int_0^l \cos \left( \frac{n\pi}{l} x \right) f(x, s) \, dx & n = 1, 2, \ldots \\ \frac{1}{l} \int_0^l f(x, s) \, dx & n = 0 \end{cases} \]

and \( C_n \) is as defined above.
(b) Prove uniqueness of this solution.

**Answer:** Suppose there are two solutions \( u \) and \( v \). Let \( w = u - v \). Therefore, \( w \) is a solution of

\[
\begin{aligned}
& w_t - w_{xx} + w = 0 & 0 < x < l \\
& w(x,0) = 0 & 0 < x < l \\
& w_x(0,t) = 0 = w_x(l,t).
\end{aligned}
\]

Multiplying this equation by \( w \) and integrating, we see that

\[
\frac{1}{2} \partial_t \int_0^l w^2(x,t) \, dx = -\int_0^l (w_x^2 + w^2) \, dx \leq 0.
\]

Therefore,

\[
\int_0^l w^2(x,t) \, dx \leq \int_0^l w^2(x,0) \, dx = 0.
\]

This implies \( w(x,t) \equiv 0 \), which implies \( u \equiv v \).

5. (a) Answer true or false to the following statements.

i. Let \( \Omega \) be an open, bounded subset of \( \mathbb{R}^n \). There exists at most one smooth solution of

\[
\begin{aligned}
& \Delta u = 0 & x & \in & \Omega \\
& \partial_{\nu} u = g & x & \in & \partial \Omega.
\end{aligned}
\]

**Answer:** False. Only unique up to a constant.

ii. If \( u \) is a bounded, harmonic function on \( \mathbb{R}^n \), then \( u \) must be constant.

**Answer:** True. Liouville’s Theorem.

iii. There exists at most one smooth, bounded solution of

\[
\begin{aligned}
& u_t - k \Delta u = f & x & \in & \mathbb{R}^n, t > 0 \\
& u(x,0) = \phi(x)
\end{aligned}
\]

**Answer:** True. Uniqueness of solutions to the heat equation on \( \mathbb{R}^n \) which satisfy the growth estimate \( u(x,t) \leq A e^{a|x|^2} \).

iv. If \( u \) is a smooth, harmonic function in \( \Omega \) and \( u = g \geq 0 \) on \( \partial \Omega \), then \( u > 0 \) in \( \Omega \).

**Answer:** False. It only guarantees \( u \geq 0 \). For example, if \( g \equiv 0 \), then \( u \equiv 0 \) in \( \Omega \).

(b) Answer the following short answer questions.

i. What is the derivative of the delta function?

**Answer:** The delta function is the distribution defined as follows.

\[
(\delta_0, \phi) = \phi(0).
\]

The derivative of the delta function is defined to be the distribution \( G \) such that

\[
(G, \phi) = -(\delta_0, \phi') = -\phi'(0).
\]
Therefore, $\delta'_0$ is the distribution such that

$$\langle \delta'_0, \phi \rangle = -\phi'(0).$$

ii. State the strong maximum principle for the heat equation.

**Answer:** Let $\Omega_T = \Omega \times (0, T]$. Let $\Gamma_T = \overline{\Omega_T} - \Omega_T$. The strong maximum principle says that if $\Omega$ is a connected set and there exists a point $(x_0, t_0) \in \Omega_T$ such that

$$u(x_0, t_0) = \max_{\Omega_T} u(x, t),$$

then $u$ must be constant.