Math 220B Midterm Exam Solutions Winter 2002

1. Use the Fourier transform to solve

$$\begin{cases} u_t = ku_{xx} - u & x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

You may use the fact that for $x \in \mathbb{R}$,

$$f(x) = e^{-\epsilon|x|^2} \implies \widehat{f}(\xi) = \frac{1}{\sqrt{2\epsilon}} e^{-|\xi|^2/4\epsilon}.$$

Answer:

$$\begin{aligned}
&u_t - ku_{xx} + u = 0 \\
&\implies \widehat{u}_t - k\widehat{u_{xx}} + \widehat{u} = 0 \\
&\implies \widehat{u}_t - k(i\xi)^2\widehat{u} + \widehat{u} = 0 \\
&\implies \widehat{u}_t + k(\xi^2 + 1)\widehat{u} = 0 \\
&\implies \frac{\widehat{u}_t}{\widehat{u}} = -(k\xi^2 + 1) \\
&\implies \ln \widehat{u} = -(k\xi^2 + 1)t + C \\
&\implies \widehat{u} = Ce^{-(k\xi^2+1)t}.
\end{aligned}$$

The initial condition implies $\widehat{u}(\xi) = \widehat{\phi} e^{-(k\xi^2+1)t}$.

Using the inverse Fourier transform, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{u}(\xi, t)\, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{\phi}(\xi) e^{-(k\xi^2+1)t}\, d\xi \\
&= \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\phi}(\xi) e^{-k\xi^2 t} e^{ix\xi}\, d\xi \\
&= \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi} \phi(y)\, dy \right] e^{-k\xi^2 t} e^{ix\xi}\, d\xi \\
&= \frac{e^{-t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(y-x)\xi} e^{-k\xi^2 t}\, d\xi \right] \phi(y)\, dy.
\end{aligned}$$

Now

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(y-x)\xi} e^{-\xi^2 t}\, d\xi = \widehat{f}(y - x)$$

where

$$f(\xi) = e^{-\xi^2 t}.$$

Now using the fact above, we see that

$$f(\xi) = e^{-k\xi^2 t} \implies \widehat{f}(x) = \frac{1}{\sqrt{2kt}} e^{-x^2/4kt}.$$

Therefore,

$$\boxed{u(x,t) = \frac{e^{-t}}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy.}$$

2. Show that $u(x) = e^{-|x|}$ solves

$$-u_{xx} + u = 2\delta_0$$

in the sense of distributions.

Answer: To say that $-u_{xx} + u = 2\delta_0$ in the sense of distributions, means the following. Let F_u be the distribution associated with u. That is, $F_u : \mathcal{D} \to \mathbb{R}$ such that

$$(F_u, \phi) = \int_{-\infty}^{\infty} e^{-|x|} \phi(x)\, dx$$

for all $\phi \in \mathcal{D}$. We need to show that

$$([-\partial_x^2 + I] F_u, \phi) = 2(\delta_0, \phi) = 2\phi(0).$$

Recall that the derivative of a distribution F is defined to be the distribution G such that

$$(G, \phi) \equiv -(F, \phi') \qquad \forall \phi \in \mathcal{D}.$$

Therefore,

$$([-\partial_x^2 + I] F_u, \phi) \equiv (F_u, [-\partial_x^2 + I]\phi).$$

Therefore, it remains to show that

$$(F_u, [-\partial_x^2 + I]\phi) = 2\phi(0) \qquad \forall\, \phi \in \mathcal{D}.$$

We proceed as follows.

$$\begin{aligned}
(F_u, [-\partial_x^2 + I]\phi) &= \int_{-\infty}^{\infty} e^{-|x|} [-\phi''(x) + \phi(x)]\, dx \\
&= \int_0^{\infty} e^{-x} [-\phi''(x)]\, dx + \int_{-\infty}^{0} e^{x} [-\phi''(x)]\, dx + \int_{-\infty}^{\infty} e^{-|x|} \phi(x)\, dx \\
&\equiv A_1 + A_2 + A_3.
\end{aligned}$$

Now integrating A_1 by parts, we see

$$\begin{aligned}
\int_0^{\infty} e^{-x} [-\phi''(x)]\, dx &= -\int_0^{\infty} e^{-x} \phi'(x)\, dx - e^{-x}\phi'(x)\Big|_0^{\infty} \\
&= -\int_0^{\infty} e^{-x} \phi(x)\, dx - e^{-x}\phi(x)\Big|_0^{\infty} - e^{-x}\phi'(x)\Big|_0^{\infty} \\
&= -\int_0^{\infty} e^{-x} \phi(x)\, dx + \phi(0) + \phi'(0).
\end{aligned}$$

Similarly, integrating A_2 by parts, we see

$$\begin{aligned}\int_{-\infty}^{0} e^x[-\phi''(x)]\,dx &= \int_{-\infty}^{0} e^x\phi'(x)\,dx - e^x\phi'(x)|_{-\infty}^{0} \\ &= -\int_{-\infty}^{0} e^x\phi(x)\,dx + e^x\phi(x)|_{-\infty}^{0} - e^x\phi'(x)|_{-\infty}^{0} \\ &= -\int_{-\infty}^{0} e^x\phi(x)\,dx + \phi(0) - \phi'(0).\end{aligned}$$

Now adding these terms with A_3, we see that

$$(F_u, [-\partial_x^2 + I]\phi) = 2\phi(0),$$

as claimed.

3. (a) Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi\}$. Solve the following boundary-value problem,

$$\begin{cases} \Delta u = 0 & (x,y) \in \Omega \\ u(0,y) = 0 = u(\pi,y) & 0 < y < \pi \\ u(x,0) = 0 & 0 < x < \pi \\ u(x,\pi) = 2\sin(x) - \sin(2x) & 0 < x < \pi. \end{cases}$$

Answer: We use separation of variables. Look for a solution of the form $u(x,y) = X(x)Y(y)$. Plugging this into the PDE, we have

$$X''Y + XY'' = 0.$$

Dividing by XY, we have

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Given that we have homogeneous boundary conditions at $x = 0$ and $x = \pi$, we will write the above equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

We are led to the eigenvalue problem,

$$\begin{aligned} &X'' = -\lambda X \qquad 0 < x < \pi \\ &X(0) = 0 = X(\pi). \end{aligned}$$

We know the eigenvalues and eigenfunctions of this problem are given by

$$\lambda_n = n^2 \qquad X_n(x) = \sin(nx) \qquad \text{for } n = 1, 2, \ldots$$

Now we need to solve our equation for Y_n for each λ_n. In particular, we need to solve

$$Y_n'' - n^2 Y_n = 0.$$

The solutions of this second-order ODE are given by

$$Y_n(y) = A_n \cosh(ny) + B_n \sinh(ny).$$

The boundary condition $u(x, 0) = 0$ implies $Y(0) = 0$. Therefore,

$$Y_n(0) = A_n = 0.$$

Therefore, for each $n = 1, 2, \ldots$, we have found a function $u_n(x, y) = X_n(x)Y_n(y)$ which is harmonic in Ω and satisfies three of the boundary conditions. It remains to satisfy our fourth boundary condition. Let

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} u_n(x, y) \\ &= \sum_{n=1}^{\infty} B_n \sinh(ny) \sin(nx) \end{aligned}$$

for B_n arbitrary. Now, we will choose B_n to satisfy our fourth boundary condition. We want $u(x, \pi) = 2\sin(x) - \sin(2x)$. That is, we want to choose B_n such that

$$\sum_{n=1}^{\infty} B_n \sinh(n\pi) \sin(nx) = 2\sin(x) - \sin(2x).$$

We note that the right-hand side is a finite combination of sine functions. Therefore, we just need to choose B_n such that

$$\begin{aligned} &B_1 \sinh(\pi) = 2 \\ &B_2 \sinh(2\pi) = -1 \\ &B_n = 0 \quad n = 3, 4, \ldots \end{aligned}$$

Therefore, our solution is given by

$$\boxed{u(x, y) = \frac{2}{\sinh(\pi)} \sinh(y) \sin(x) - \frac{1}{\sinh(2\pi)} \sinh(2y) \sin(2x).}$$

(b) Find $\max_{\overline{\Omega}} u(x, y)$.

Answer: We know from the maximum principle for harmonic functions that

$$\max_{\overline{\Omega}} u(x, y) = \max_{\partial\Omega} u(x, y).$$

Now, $u(x, y) = 0$ along three of the sides. We just need to check what the maximum of u is on the fourth side. On the fourth side,

$$u(x, \pi) = 2\sin(x) - \sin(2x).$$

To find the maximum on this side, we look for critical points of $u(x,\pi)$.

$$\begin{aligned} u_x(x,\pi) = 2\cos(x) - 2\cos(2x) &= 0 \\ \implies \cos(x) &= \cos(2x) \\ \implies \cos(x) &= 2\cos^2(x) - 1 \\ \implies (2\cos(x)+1)(\cos(x)-1) &= 0 \\ \implies x &= 0, 2\pi/3. \end{aligned}$$

We see that $u(0,\pi) = 0$. We see that $u(2\pi/3,\pi) = 2\sin(2\pi/3) - \sin(4\pi/3) = 2\sqrt{3}/2 + \sqrt{3}/2 = 3\sqrt{3}/2$. Therefore,

$$\boxed{\max_{\overline{\Omega}} u(x,y) = 3\sqrt{3}/2}.$$

4. (a) Solve

$$\begin{cases} u_t - u_{xx} + u = f(x) & 0 < x < l \\ u(x,0) = g(x) & 0 < x < l \\ u_x(0,t) = 0 = u_x(l,t). \end{cases}$$

Answer: First, we will solve the homogeneous problem

$$u_t - u_{xx} + u = 0.$$

Using separation of variables, we are led to the eigenvalue problem

$$\begin{cases} X'' = (1-\lambda)X & 0 < x < l \\ X'(0) = 0 = X'(l). \end{cases}$$

Letting $\mu = \lambda - 1$, this becomes the eigenvalue problem

$$\begin{cases} X'' = -\mu X & 0 < x < l \\ X'(0) = 0 = X'(l). \end{cases}$$

We know the eigenfunctions and eigenvalues for this problem are

$$X_n(x) = \cos\left(\frac{n\pi}{l}x\right) \quad \mu_n = \left(\frac{n\pi}{l}\right)^2 \qquad n = 0, 1, 2, \ldots$$

Therefore, $\lambda_n = 1 + \mu_n = 1 + \left(\frac{n\pi}{l}\right)^2$. Now we need to solve the equation for T_n,

$$T_n' = -\lambda_n T \implies T_n(t) = C_n e^{-(1+(n\pi/l)^2)t}.$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) e^{-(1+(n\pi/l)^2)t}.$$

In order for our initial condition to be satisfied, we need

$$u(x,0) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) = g(x).$$

This is just the Fourier cosine series for ϕ. The coefficients C_n are given by

$$C_n = \frac{\langle \cos\left(\frac{n\pi}{l}x\right), g(x)\rangle}{\langle \cos\left(\frac{n\pi}{l}x\right), \cos\left(\frac{n\pi}{l}x\right)\rangle} = \frac{\int_0^l \cos\left(\frac{n\pi}{l}x\right) g(x)\,dx}{\int_0^l \cos^2\left(\frac{n\pi}{l}x\right)\,dx}.$$

Therefore,

$$C_n = \begin{cases} \frac{2}{l}\int_0^l \cos\left(\frac{n\pi}{l}x\right) g(x)\,dx & n = 1, 2, \ldots \\ \frac{1}{l}\int_0^l g(x)\,dx & n = 0. \end{cases}$$

Therefore, the solution operator $S(t)$ associated with the homogeneous equation is the operator such that

$$S(t)\phi = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) e^{-(1+(n\pi/l)^2)t}$$

with C_n as defined above. Using Duhamel's principle, we know that the solution of the inhomogeneous equation is given by

$$u(x,t) = S(t)\phi + \int_0^t S(t-s)f(s)\,ds.$$

Therefore, the solution of our problem is given by

$$\boxed{\begin{aligned} u(x,t) = &\sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) e^{-(1+(n\pi/l)^2)t} \\ &+ \int_0^t \sum_{n=0}^{\infty} D_n(s) \cos\left(\frac{n\pi}{l}x\right) e^{-(1+(n\pi/l)^2)(t-s)}\,ds \end{aligned}}$$

where

$$\boxed{C_n = \begin{cases} \frac{2}{l}\int_0^l \cos\left(\frac{n\pi}{l}x\right) g(x)\,dx & n = 1, 2, \ldots \\ \frac{1}{l}\int_0^l \phi(x)\,dx & n = 0. \end{cases}}$$

and

$$\boxed{D_n(s) = \begin{cases} \frac{2}{l}\int_0^l \cos\left(\frac{n\pi}{l}x\right) f(x,s)\,dx & n = 1, 2, \ldots \\ \frac{1}{l}\int_0^l f(x,s)\,dx & n = 0 \end{cases}}$$

and C_n is as defined above.

(b) Prove uniqueness of this solution.

Answer: Suppose there are two solutions u and v. Let $w = u - v$. Therefore, w is a solution of

$$\begin{cases} w_t - w_{xx} + w = 0 & 0 < x < l \\ w(x,0) = 0 & 0 < x < l \\ w_x(0,t) = 0 = w_x(l,t). \end{cases}$$

Multiplying this equation by w and integrating, we see that

$$\frac{1}{2}\partial_t \int_0^l w^2(x,t)\,dx = -\int_0^l (w_x^2 + w^2)\,dx \leq 0.$$

Therefore,

$$\int_0^l w^2(x,t)\,dx \leq \int_0^l w^2(x,0)\,dx = 0.$$

This implies $w(x,t) \equiv 0$, which implies $u \equiv v$.

5. (a) Answer true or false to the following statements.

   i. Let Ω be an open, bounded subset of $\mathbb{R}^n$. There exists at most one smooth solution of

   $$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega. \end{cases}$$

   Answer: False. Only unique up to a constant.

   ii. If u is a bounded, harmonic function on $\mathbb{R}^n$, then u must be constant.
   Answer: True. Liouville's Theorem.

   iii. There exists at most one smooth, bounded solution of

   $$\begin{cases} u_t - k\Delta u = f & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

   Answer: True. Uniqueness of solutions to the heat equation on $\mathbb{R}^n$ which satisfy the growth estimate $u(x,t) \leq Ae^{a|x|^2}$.

   iv. If u is a smooth, harmonic function in Ω and $u = g \geq 0$ on $\partial\Omega$, then $u > 0$ in Ω.
   Answer: False. It only guarantees $u \geq 0$. For example, if $g \equiv 0$, then $u \equiv 0$ in Ω.

(b) Answer the following short answer questions.

   i. What is the derivative of the delta function?
   Answer: The delta function is the distribution defined as follows.

   $$(\delta_0, \phi) = \phi(0).$$

   The derivative of the delta function is defined to be the distribution G such that

   $$(G, \phi) = -(\delta_0, \phi') = -\phi'(0).$$

Therefore, δ_0' is the distribution such that

$$\boxed{(\delta_0', \phi) = -\phi'(0).}$$

ii. State the strong maximum principle for the heat equation.
Answer: Let $\Omega_T = \Omega \times (0, T]$. Let $\Gamma_T \equiv \overline{\Omega_T} - \Omega_T$. The strong maximum principle says that if Ω is a connected set and there exists a point $(x_0, t_0) \in \Omega_T$ such that

$$u(x_0, t_0) = \max_{\overline{\Omega_T}} u(x, t),$$

then u must be constant.