Comments on Rayleigh-Ritz Approximation and Minimax Principle

Let $\{w_1, \ldots, w_n\}$ be n linearly independent trial functions. Let $A = (a_{jk})$, $B = (b_{jk})$ where $a_{jk} = \langle \nabla w_j, \nabla w_k \rangle$ and $b_{jk} = \langle w_j, w_k \rangle$. Consider the equation

$$\det(A - \lambda B) = 0.$$

Assume $\lambda_1^*, \ldots, \lambda_n^*$ are the n real roots of this equation. We know that there exists a set of vectors $\{v_j\}$ which form a basis for $\mathbb{R}^n$ such that

$$Av_j = \lambda_j^* B v_j \qquad j = 1, \ldots, n$$

and the $\{v_j\}$ are mutually orthogonal with respect to B; that is,

$$Bv_j \cdot v_i = 0 \qquad i \neq j.$$

Let

$$X_i \equiv \text{span}\{v_1, \ldots v_i\}.$$

Lemma 1. *For $\{w_1, \ldots, w_n\}$ a set of linearly independent trial functions and A, B, $\{v_j\}$ defined as above, the i^{th} root of the equation $\det(A - \lambda B) = 0$ satisfies*

$$\lambda_i^* = \max_{c \in X_i, c \neq 0} \frac{Ac \cdot c}{Bc \cdot c}$$

Proof. First, we note that

$$\begin{aligned} \lambda_i^* &= \frac{Av_i \cdot v_i}{Bv_i \cdot v_i} \\ &\leq \max_{c \in X_i} \frac{Ac \cdot c}{Bc \cdot c}. \end{aligned}$$

Next, let $c \in X_i$. Then $c = \sum_{j=1}^{i} c_j v_j$. Therefore,

$$\begin{aligned} \frac{Ac \cdot c}{Bc \cdot c} &= \frac{A \sum_{j=1}^{i} c_j v_j \cdot \sum_{j=1}^{i} c_j v_j}{B \sum_{j=1}^{i} c_j v_j \cdot \sum_{j=1}^{i} c_j v_j} \\ &= \frac{\sum_{j=1}^{i} c_j \lambda_j^* B v_j \cdot \sum_{j=1}^{i} c_j v_j}{\sum_{j=1}^{i} c_j^2 B v_j \cdot v_j} \\ &= \frac{\sum_{j=1}^{i} c_j^2 \lambda_j^* B v_j \cdot v_j}{\sum_{j=1}^{i} c_j^2 B v_j \cdot v_j} \\ &\leq \lambda_i^*. \end{aligned}$$

Therefore, taking the maximum of both sides over all possible $c \in X_i$, we get the desired result. □

Remark. Using the fact that

$$\frac{Ac \cdot c}{Bc \cdot c} = \frac{||\nabla(\sum_{j=1}^n c_j w_j)||_{L^2}^2}{||\sum_{j=1}^n c_j w_j||_{L^2}^2},$$

we have

$$\lambda_i^* = \max_{c \in X_i, c \neq 0} \left\{ \frac{||\nabla(\sum_{j=1}^n c_j w_j)||_{L^2}^2}{||\sum_{j=1}^n c_j w_j||_{L^2}^2} \right\}.$$

Lemma 2. *The i^{th} Dirichlet eigenvalue is given by*

$$\lambda_i = \min \max_{c \in X_i, c \neq 0} \left\{ \frac{||\nabla(\sum_{j=1}^n c_j w_j)||_{L^2}^2}{||\sum_{j=1}^n c_j w_j||_{L^2}^2} \right\}$$

where the minimum is taken over all possible sets of linearly independent functions $\{w_1, \ldots, w_n\}$.

Remark: From the above remark, we see that

$$\lambda_i = \min \lambda_i^*(w_1, \ldots, w_n)$$

where the minimum is taken over all possible sets of linearly independent trial functions.

Proof. Fix $\{w_1, \ldots, w_n\}$. Choose a linear combination $w = \sum_{j=1}^n c_j w_j$ such that

- $c = (c_1, \ldots, c_n) \in X_i$
- w is orthogonal to the first $i-1$ eigenfunctions (denoted u_i).

The first condition implies

$$c \cdot Bv_j = 0 \quad j = i+1, \ldots, n.$$

The second condition implies

$$\langle w, u_j \rangle = 0 \quad j = 1, \ldots, i-1.$$

In particular, we have $n-1$ equations for our n unknowns $c_1, \ldots, c_n$. Therefore, such a function exists.

By the Minimum Principle for the i^{th} eigenvalue, we have

$$\begin{aligned} \lambda_i &\leq \frac{||\nabla w||_{L^2}^2}{||w||_{L^2}^2} \\ &\leq \max_{c \in X_i} \frac{||\nabla(\sum_{j=1}^n c_j w_j)||_{L^2}^2}{||\sum_{j=1}^n c_j w_j||_{L^2}^2}. \end{aligned}$$

Taking the minimum of both sides over all possible sets of linearly independent trial functions, we conclude that

$$\lambda_i \leq \min \max_{c \in X_i, c \neq 0} \frac{||\nabla(\sum_{j=1}^n c_j w_j)||_{L^2}^2}{||\sum_{j=1}^n c_j w_j||_{L^2}^2}.$$

Next, let $\{w_1, \dots, w_n\}$ be the first n eigenfunctions. Then

$$\max_{c \in X_i} \frac{||\nabla(\sum_{j=1}^n c_j w_j)||_{L^2}^2}{||\sum_{j=1}^n c_j w_j||_{L^2}^2} = \frac{||\nabla w_i||_{L^2}^2}{||w_i||_{L^2}^2} = \lambda_i.$$

Therefore,

$$\min \max_{c \in X_i} \frac{||\nabla(\sum_{j=1}^n c_j w_j)||_{L^2}^2}{||\sum_{j=1}^n c_j w_j||_{L^2}^2} \leq \lambda_i.$$

Consequently, we get the desired result. □