Physical Interpretation of Single and Double Layer Potentials

Below is a brief description of a physical interpretation for the single and double layer potentials.

References

- Folland, G. *Introduction to Partial Differential Equations*, Chap. 3.
- McOwen, R. *Partial Differential Equations: Methods and Applications*, Chap. 4.

Single Layer Potentials

Let y be a point in $\mathbb{R}^3$ where a single unit charge has been placed. The electric field at the point x induced by the charge at point y is given by

$$\mathbf{E}(x) = \frac{x-y}{|x-y|^3}.$$

The associated *potential function* is $u(x) = \frac{1}{|x-y|}$. That is, $\mathbf{E}(x) = -\nabla_x u$. Now suppose there is a domain Ω in $\mathbb{R}^3$ with a distribution of charges. Then the potential at any point $x \in \mathbb{R}^3$ associated with the electric field generated by these charges can by written as

$$u(x) = \int_\Omega \frac{1}{|x-y|}\rho(y)\,dy$$

where ρ is the charge density in Ω. Similarly, we can talk about the potential associated with the electric field generated by a charge distribution on a surface $\partial\Omega$. This *layer potential* is denoted

$$u(x) = \int_{\partial\Omega} \frac{1}{|x-y|}\rho(y)\,dS(y).$$

We note that

$$\int_{\partial\Omega} \frac{1}{|x-y|}\rho(y)\,dS(y) = 4\pi \int_{\partial\Omega} \Phi(y-x)\rho(y)\,dS(y)$$

where Φ is the fundamental solution of Laplace's equation in $\mathbb{R}^3$. Therefore, the *single layer potential*

$$\overline{u}(x) = -\int_{\partial\Omega} \Phi(y-x)\rho(y)\,dS(y)$$

is a multiple of the potential induced by a charge distribution of density ρ on a surface $\partial\Omega$.

Double Layer Potentials

We can think of the physical interpretation of a double layer potential,

$$\overline{\overline{u}}(x) = -\int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_y}(x-y)h(y)\,dS(y),$$

as follows. Fix some $t > 0$. Suppose we have a charge distribution on a surface S in $\mathbb{R}^3$ such that the charge density at any point y on the surface is given by $t^{-1}\rho(y)$. In addition, suppose we have a charge distribution of opposite sign on the parallel surface $S_t = \{y + t\nu(y) : y \in S\}$ such that the charge density on S_t is given by $-t^{-1}\rho(y)$. Then the electric field at any point

x in $\mathbb{R}^3$ generated by these electric charges is given by $\mathbf{E}(x) = -\nabla u(x)$ where u is the associated potential, given by

$$u(x) = \int_S \left[\frac{1}{|x-y|} - \frac{1}{|x-(y+t\nu(y))|} \right] \frac{\rho(y)}{t} \, dS(y).$$

As $t \to 0$,

$$\left[\frac{1}{|x-y|} - \frac{1}{|x-(y+t\nu(y))|} \right] \frac{1}{t} \to \frac{\partial}{\partial \nu} \left(\frac{1}{|x-y|} \right).$$

Therefore, the double layer potential can be thought of as a multiple of the potential induced by a double layer of charges of opposite sign on $\partial\Omega$.