Math 220B - Summer 2003
Homework 1 Solutions

1. Consider the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X \text{ satisfies symmetric B.C.s.} & x = 0, l. \end{cases}$$

Suppose

$$f(x)f'(x)\Big|_{x=a}^{x=b} \leq 0$$

for all real-valued functions $f(x)$ which satisfy the boundary conditions. Show there are no negative eigenvalues.

Answer: Let X be an eigenfunction with eigenvalue λ.

$$\begin{aligned} \lambda\langle X, X\rangle &= -\langle X'', X\rangle \\ &= \langle X', X'\rangle - X'X\Big|_{x=0}^{x=l} \\ &\geq 0 \end{aligned}$$

using the assumption that $X'X|_{x=0}^{x=l} \leq 0$.

2. Consider the eigenvalue problem,

$$\begin{cases} X'' = -\lambda X & a < x < b \\ X \text{ satisfies certain B.C.'s.} \end{cases}$$

Suppose μ is an eigenvalue of multiplicity $m > 1$. Let $X_1, \ldots, X_m$ denote m linearly independent eigenfunctions (which may or may not be orthogonal) associated with the eigenvalue μ. Use these eigenfunctions to construct m eigenfunctions $Y_1, \ldots, Y_m$ which are necessarily orthogonal.

Answer: First, define

$$Y_1 = \frac{X_1}{||X_1||}.$$

Next, we define a function orthogonal to Y_1 by subtracting the piece of X_2 which is parallel to Y_1. Therefore, let

$$Z_2 = X_2 - \langle X_2, Y_1\rangle Y_1.$$

We normalize, letting $Y_2 = Z_2/||Z_2||$. Continuing in this way, we let

$$Z_3 = X_3 - \langle X_3, Y_1\rangle Y_1 - \langle X_3, Y_2\rangle Y_2$$

and then define $Y_3 = Z_3/||Z_3||$. In general, we define

$$Z_m = X_m - \langle X_m, Y_1\rangle Y_1 - \ldots - \langle X_m, Y_{m-1}\rangle Y_{m-1},$$

and let $Y_m = Z_m/||Z_m||$.

3. Consider the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X'(0) + X(0) = 0 \\ X(l) = 0. \end{cases}$$

(a) Find an equation for the positive eigenvalues.

Answer: If $\lambda = \beta^2 > 0$, then

$$X'' + \beta^2 X = 0 \implies X(x) = A\cos(\beta x) + B\sin(\beta x).$$

The boundary condition

$$X'(0) + X(0) = B\beta + A = 0.$$

The boundary condition

$$X(l) = 0 \implies A\cos(\beta l) + B\sin(\beta l) = 0.$$

Combining these two equations, we conclude that λ is a positive eigenvalue if and only if

$$-\beta\cos(\beta l) + \sin(\beta l) = 0$$

or

$$\boxed{\tan(\beta l) = \beta.}$$

(b) Show graphically that there are an infinite number of positive eigenvalues.

Answer: Let

$$\begin{aligned} f(\beta) &= \beta \\ g(\beta) &= \tan(\beta l). \end{aligned}$$

We know that $g(\beta)$ has vertical asymptotes at $\beta = \left(n + \frac{1}{2}\right)\pi/l$ and $g(\beta) \to +\infty$ as $\beta \to \left(n + \frac{1}{2}\right)\pi/l$. By graphing both functions, we see that $f(\beta)$ and $g(\beta)$ have an infinite number of intersections.

(c) Show that $\lambda = 0$ is an eigenvalue if and only if $l = 1$. Find a corresponding eigenfunction in this case.

Answer:

$$\lambda = 0 \implies X'' = 0 \implies X(x) = A + Bx.$$

The boundary condition

$$X'(0) + X(0) = 0 \implies B + A = 0.$$

The boundary condition

$$X(l) = 0 \implies A + Bl = 0.$$

Combining these two equations implies

$$B(1-l) = 0.$$

Therefore, zero is an eigenvalue if and only if $l = 1$. In this case, the corresponding eigenfunction is

$$\boxed{X_0(x) = B_0(1-x)}$$

for any B_0.

(d) Show that if $l \leq 1$, then there are no negative eigenvalues, but if $l > 1$, then there is one negative eigenvalue. Find the corresponding eigenfunction.

Answer:

$$\lambda = -\gamma^2 < 0 \implies X(x) = A\cosh(\gamma x) + B\sinh(\gamma x).$$

The boundary condition

$$X'(0) + X(0) = 0 \implies B\gamma + A = 0.$$

The boundary condition

$$X(l) = 0 \implies A\cosh(\gamma l) + B\sinh(\gamma l) = 0.$$

Combining these two equations, we conclude that

$$-B\gamma\cosh(\gamma l) + B\sinh(\gamma l) = 0.$$

If $B = 0$, then $X(x) \equiv 0$. Therefore, we must have

$$-\gamma\cosh(\gamma l) + \sinh(\gamma l) = 0,$$

or

$$\tanh(\gamma l) = \gamma.$$

To look for roots of this equation, we consider the graphs of the functions

$$\begin{aligned} f(\gamma) &= \gamma \\ g(\gamma) &= \tanh(\gamma l). \end{aligned}$$

We note that $f'(\gamma) = 1$ and $g'(\gamma) = l\operatorname{sech}^2(\gamma l)$. Therefore, $f'(0) = 1$, $g'(0) = l$. Further, we note that $g''(\gamma) = -2l\operatorname{sech}^2(\gamma l)\tanh(\gamma l) < 0$. Also, we note that $g(\gamma) \to 1$ as $\gamma \to +\infty$.

From this information, we conclude that $f(\gamma)$ and $g(\gamma)$ only intersect at $\gamma = 0$ if $l \leq 1$, and, consequently, there are no negative eigenvalues if $l \leq 1$.

On the other hand, if $l > 1$, $f(\gamma)$ and $g(\gamma)$ will intersect once, implying there is exactly one negative eigenvalue if $l > 1$.

4. Use separation of variables to solve

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \\ u_x(l,t) = 0. \end{cases}$$

Answer: We use separation of variables, letting $u(x,t) = X(x)T(t)$. Plugging this function into our PDE, we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

We are led to the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X(0) = 0 = X'(l). \end{cases}$$

Looking for positive eigenvalues,

$$\lambda = \beta^2 > 0 \implies X(x) = A\cos(\beta x) + B\sin(\beta x).$$

The boundary condition

$$X(0) = 0 \implies A = 0.$$

The boundary condition

$$X'(l) = 0 \implies B\beta\cos(\beta l) = 0 \implies \beta_n = \frac{\left(n + \frac{1}{2}\right)\pi}{l}.$$

Therefore,

$$\lambda_n = \left(\frac{\left(n + \frac{1}{2}\right)\pi}{l}\right)^2 \qquad X_n(x) = B_n \sin\left(\frac{\left(n + \frac{1}{2}\right)\pi}{l} x\right).$$

By straightforward calculation, we see these are all the eigenvalues. Solving the equation for T_n,

$$\begin{aligned} T' = -k\lambda_n T &\implies T_n(t) = C_n e^{-k\lambda_n t} \\ &\implies T_n(t) = C_n e^{-k\left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2 t}. \end{aligned}$$

Therefore, we let

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{\left(n + \frac{1}{2}\right)\pi}{l} x\right) e^{-k\left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2 t}.}$$

In order for the initial condition $u(x,0) = \phi(x)$ to be satisfied, we need

$$\boxed{C_n = \frac{\langle \sin\left(\frac{(n+\frac{1}{2})\pi}{l} x\right), \phi(x)\rangle}{\langle \sin\left(\frac{(n+\frac{1}{2})\pi}{l} x\right), \sin\left(\frac{(n+\frac{1}{2})\pi}{l} x\right)\rangle}.}$$

5. Use separation of variables to solve

$$\begin{cases} u_t - ku_{xx} + u_x = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \\ u_x(0,t) = 0. \end{cases}$$

(*Hint:* Introduce a function $f(x)$ such that $v(x,t) = f(x)u(x,t)$ will satisfy a PDE of the form

$$v_t - kv_{xx} + av = 0$$

with new initial and boundary conditions. Solve the equation for v, and from this solution, solve for u.)

Answer: We will give two methods.

Method 1. We want to find a function f such that defining $v = fu$, then v satisfies a PDE of the form $v_t - kv_{xx} + av = 0$. Let $g = 1/f$. Therefore, $u = gv$. Plugging this function into our PDE, we have

$$gv_t - kgv_{xx} + [-2kg' + g]v_x + [-kg'' + g']v = 0.$$

In order to eliminate the v_x term, we want to choose g such that $-2kg' + g = 0$, which implies $g(x) = e^{x/2k}$. Then let $v = fu = (1/g)u = e^{-x/2k}u$. Solving for $-kg'' + g'$, we see that v will satisfy the following initial/boundary-value problem

$$\begin{cases} v_t - kv_{xx} + \frac{1}{4k}v = 0 & 0 < x < l, t > 0 \\ v(x,0) = e^{-x/2k}\phi(x) \\ v(0,t) = 0 \\ v_x(l,t) + \frac{1}{2k}v(l,t) = 0. \end{cases}$$

Now use separation of variables on this initial/boundary-value problem. Plugging in a function $v(x,t) = X(x)T(t)$ implies

$$\frac{T'}{kT} = \frac{X''}{X} - \frac{1}{4k} = -\lambda.$$

We need to solve the ODE

$$\frac{X''}{X} = -\lambda + \frac{1}{4k}.$$

Let $-\mu = -\lambda + \frac{1}{4k}$. We are led to the eigenvalue problem

$$\begin{cases} X'' = -\mu X & 0 < x < l \\ X(0) = 0 \\ X'(l) + \frac{1}{2k}X(l) = 0. \end{cases}$$

If $\mu = \beta^2 > 0$, then $X(x) = A\cos(\beta x) + B\sin(\beta x)$. The boundary condition

$$X(0) = 0 \implies A = 0.$$

The boundary condition

$$X'(l) + \frac{1}{2k}X(l) = 0 \implies B\beta\cos(\beta l) + \frac{1}{2k}B\sin(\beta l) = 0.$$

If $B = 0$, then $X(x) \equiv 0$. Therefore, we need

$$\beta\cos(\beta l) + \frac{1}{2k}\sin(\beta l) = 0,$$

or

$$\tan(\beta l) = -2k\beta.$$

The corresponding eigenfunctions are

$$X_n(x) = B_n \sin(\beta_n x).$$

If $\mu = 0$, then $X(x) = A + Bx$. The boundary condition

$$X(0) = 0 \implies A = 0.$$

The boundary condition

$$X'(l) + \frac{1}{2k}X(l) = 0 \implies B + \frac{1}{2k}Bl = 0 \implies B = 0.$$

Therefore, zero is not an eigenvalue.

Using the result from problem (1), we can conclude that there are no negative eigenvalues.

Recalling the definition of μ, we see that

$$\boxed{\lambda_n = \beta_n^2 + \frac{1}{4k}}$$

where

$$\boxed{\tan(\beta_n l) = -2k\beta_n.}$$

Solving our equation for T_n, we have

$$T_n(t) = C_n e^{-k\lambda_n t}.$$

Therefore, we let

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} C_n \sin(\beta_n x) e^{-k\lambda_n t}}$$

where λ_n and β_n are defined above. We want $v(x,0) = e^{-x/2k}\phi(x)$. Therefore, let

$$\boxed{C_n = \frac{\langle \sin(\beta_n x), e^{-x/2k}\phi(x)\rangle}{\langle \sin(\beta_n x), \sin(\beta_n x)\rangle}.}$$

Finally, we recall that $\boxed{u(x,t) = e^{x/2k} v(x,t)}$ for v defined above.

Method 2. Here is an alternate method, which allows one to use separation of variables directly, but requires knowledge about Sturm-Liouville problems. (You are not responsible for this material, but I thought I'd write up a solution using this method because some students used this technique.)

Let $u = XT$. Plugging u into our equation, we have

$$\frac{T'}{kT} = \frac{X'' - \frac{1}{k}X'}{X} = -\lambda.$$

We need to consider the eigenvalue problem

$$\begin{cases} -X'' + \frac{1}{k}X' - \lambda X = 0 & 0 < x < l \\ X(0) = 0 = X'(l). \end{cases}$$

Multiplying by the integrating factor $e^{-x/k}$, we see this ODE can be rewritten as

$$-\left(e^{-x/k}X'\right)' - \lambda e^{-x/k} X = 0,$$

an example of a Sturm-Liouville problem. We note that eigenfunctions of this Sturm-Liouville problem are orthogonal with respect to the weight function $e^{-x/k}$.

To find the solutions of our second-order ODE,

$$X'' - \frac{1}{k}X' + \lambda X = 0,$$

we look at the characteristic polynomial $p(r) = r^2 - \frac{1}{k}r + \lambda = 0$. The roots of this polynomial are given by

$$\begin{aligned} r &= \frac{\frac{1}{k} \pm \sqrt{\frac{1}{k^2} - 4\lambda}}{2} \\ &= \frac{1}{2k} \pm \sqrt{\frac{1}{4k^2} - \lambda}. \end{aligned}$$

Let $-\mu = \frac{1}{4k^2} - \lambda$. Then

$$r = \frac{1}{2k} \pm \sqrt{-\mu}.$$

The first case is $\mu > 0$. In this case,

$$r = \frac{1}{2k} \pm i\sqrt{\mu},$$

which implies the associated eigenfunctions are given by

$$X(x) = e^{x/2k}[A\cos(\sqrt{\mu}x) + B\sin(\sqrt{\mu}x)].$$

The boundary condition

$$X(0) = 0 \implies A = 0.$$

The boundary condition

$$X'(l) = 0 \implies [\frac{1}{2k}\sin(\sqrt{\mu}l) + \sqrt{\mu}\cos(\sqrt{\mu}l)] = 0 \implies \tan(\sqrt{\mu}l) = -2k\sqrt{\mu}.$$

Thus far, we have eigenvalues

$$\lambda_n = \frac{1}{4k^2} + \mu_n$$

where

$$\tan(\sqrt{\mu}_n l) = -2k\sqrt{\mu}_n$$

and the associated eigenfunctions

$$X_n(x) = e^{x/2k}\sin(\sqrt{\mu}_n x).$$

Next, we consider $\mu = 0$. In this case, $r = \frac{1}{2k}$ which implies

$$X(x) = Ae^{x/2k} + Bxe^{x/2k}.$$

The boundary condition

$$X(0) = 0 \implies A = 0.$$

The boundary condition

$$X'(l) = 0 \implies B\left[e^{x/2l} + \frac{1}{2k}le^{l/2k}\right] = 0.$$

But this can only happen if $B = 0$ which implies $X(x) \equiv 0$. Therefore, $\mu \neq 0$.

Next, we consider $\mu < 0$. In this case, $r = \frac{1}{2k} \pm \sqrt{-\mu}$, which implies

$$X(x) = e^{x/2k}\left[Ae^{\sqrt{-\mu x}} + Be^{-\sqrt{-\mu x}}\right].$$

The boundary condition

$$X(0) = 0 \implies A + B = 0.$$

The boundary condition

$$X'(l) = 0 \implies \left(\frac{\frac{1}{2k} + \sqrt{-\mu}}{\frac{1}{2k} - \sqrt{-\mu}}\right)e^{2\sqrt{-\mu}l} = 1$$

or $B = 0$. Since both terms in the expression above are greater than 1, we must have $B = 0$. Therefore, $\mu \nless 0$.

We conclude that

$$\boxed{\lambda = \frac{1}{4k^2} + \mu}$$

where $\mu > 0$ such that

$$\boxed{\tan(\sqrt{\mu} l) = -2k\sqrt{\mu}}$$

and

$$\boxed{X_n(x) = e^{x/2k} \sin(\sqrt{\mu}_n x).}$$

The solution for our equation for T is given by

$$\boxed{T_n(t) = C_n e^{-k\lambda_n t}.}$$

Therefore, let

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n e^{x/2k} \sin(\sqrt{\mu}_n x) e^{-k\lambda_n t}.}$$

We want $u(x,0) = \phi(x)$. That is, we want to choose C_n such that

$$\sum_{n=1}^{\infty} C_n e^{x/2k} \sin(\sqrt{\mu}_n x) = \phi(x).$$

Using the fact that eigenfunctions are orthogonal with respect to the weight function $e^{-x/2k}$, we can multiply both sides by $\sin(\sqrt{\mu}_m x)$ and integrate over $[0,l]$ with respect to the weight function $e^{-x/2k}$.

In particular, we conclude that

$$C_n = \frac{\int_0^l \phi(x) e^{x/2k} \sin(\sqrt{\mu}_n x) e^{-x/k} \, dx}{\int_0^l (e^{x/2k} \sin(\sqrt{\mu}_n x))^2 e^{-x/k} \, dx}.$$

Simplifying, we have

$$\boxed{C_n = \frac{\int_0^l \phi(x) e^{-x/2k} \sin(\sqrt{\mu}_n x) \, dx}{\int_0^l \sin^2(\sqrt{\mu}_n x) \, dx}.}$$