Math 220B - Summer 2003
Homework 2
Due Thursday, July 10, 2003

1. (a) Compute the Fourier transform of xf in terms of $\widehat{f}$.

   (b) Compute the Fourier transform of xe^{-tx^2}.

2. Use the Fourier transform to show that the solution of the inhomogeneous heat equation with zero initial data,

$$\begin{cases} u_t - ku_{xx} = f(x,t) & -\infty < x < \infty, t > 0 \\ u(x,0) = 0 & -\infty < x < \infty \end{cases}$$

   is given by

$$u(x,t) = \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4k(t-s)} f(y,s)\, dy\, ds.$$

3. Use the Fourier transform to solve

$$\begin{cases} u_t - tu_{xx} = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) & -\infty < x < \infty \end{cases}$$

4. (a) Consider the heat equation on a half-line with Dirichlet boundary conditions

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < \infty, t > 0 \\ u(x,0) = \phi(x) & 0 < x < \infty \\ u(0,t) = 0 & t > 0. \end{cases}$$

   Solve for $u(x,t)$.

   (b) Consider the heat equation on a half-line with Robin boundary conditions

$$(*) \begin{cases} u_t - ku_{xx} = 0 & 0 < x < \infty, t > 0 \\ u(x,0) = \phi(x) & 0 < x < \infty \\ u_x(0,t) - hu(x,t) = 0 & t > 0. \end{cases}$$

   Solve this initial value problem as follows. Assuming u is the solution of (*), introduce a new function v such that $v(x,t) = u_x(x,t) - hu(x,t)$.

   i. Determine the initial/boundary value problem that v satisfies.

   ii. Solve for u in terms of v.

5. Consider the initial/boundary-value problem

$$(**) \quad \begin{cases} u_t - ku_{xx} = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) & 0 < x < l \\ u(0,t) = 0 = u(l,t) & t > 0. \end{cases}$$

Let $\phi_{ext}(x)$ be the extension of ϕ to all of $\mathbb{R}$ such that ϕ_{ext} is odd with respect to $x = 0$ and ϕ_{ext} is $2l$-periodic. That is,

$$\phi_{ext}(x) = \begin{cases} \phi(x) & 0 < x < l \\ -\phi(-x) & -l < x < 0 \end{cases}$$

and ϕ is $2l$-periodic.

(a) Consider the initial-value problem

$$\begin{cases} v_t - kv_{xx} = 0 & -\infty < x < \infty, t > 0 \\ v(x,0) = \phi_{ext}(x) & -\infty < x < \infty. \end{cases}$$

Write the solution formula for v. Show that if $u(x,t)$ is defined to be $v(x,t)$ for $0 \leq x \leq l$, then u will satisfy (**).

(b) Assume that

$$\phi(x) = \sum_{n=1} A_n \sin\left(\frac{n\pi}{l}x\right) \text{ for } 0 \leq x \leq l,$$

where

$$A_n = \frac{\langle \phi, \sin\left(\frac{n\pi}{l}x\right)\rangle}{\langle \sin\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right)\rangle}.$$

(That is, assume that the Fourier sine series for ϕ converges to ϕ.) Note that for ϕ_{ext} defined above,

$$\phi_{ext}(x) = \sum_{n=1} A_n \sin\left(\frac{n\pi}{l}x\right) \text{ for } -\infty < x < \infty.$$

Using the solution formula found in part (a), show that

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) e^{-kn^2\pi^2 t/l^2}$$

with A_n defined above. (Consequently if $u(x,t) = v(x,t)$ for $0 \leq x \leq l$, then u has this form. In particular, we have justified the separation of variables technique.)