Math 220B - Summer 2003
Homework 2 Solutions

1. (a) Compute the Fourier transform of xf in terms of $\widehat{f}$.

   Answer:

$$\begin{aligned}
\widehat{xf}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} x f(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{-i} \frac{d}{d\xi} \left(e^{-ix\xi}\right) f(x)\, dx \\
&= \frac{1}{-i} \frac{d}{d\xi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx \right) \\
&= i \frac{d}{d\xi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx \right) \\
&= i \frac{d}{d\xi} \widehat{f}(\xi).
\end{aligned}$$

$$\boxed{\widehat{xf}(\xi) = i \frac{d}{d\xi} \widehat{f}(\xi).}$$

   (b) Compute the Fourier transform of xe^{-tx^2}.

   Answer: From part (a),

$$\widehat{xe^{-tx^2}} = i \frac{d}{d\xi} \left(\widehat{e^{-tx^2}} \right).$$

   Recall $f(x) = e^{-\epsilon x^2}$ implies $\widehat{f}(\xi) = \frac{1}{\sqrt{2\epsilon}} e^{-\xi^2/4\epsilon}$. Therefore, $\widehat{e^{-tx^2}} = \frac{1}{\sqrt{2t}} e^{-\xi^2/4t}$. Therefore,

$$\begin{aligned}
\widehat{xe^{-tx^2}} &= i \frac{d}{d\xi} \left(\frac{1}{\sqrt{2t}} e^{-\xi^2/4t} \right) \\
&= \frac{i}{\sqrt{2t}} \cdot \frac{-\xi}{2t} e^{-\xi^2/4t} \\
&= \frac{-i\xi}{(2t)^{3/2}} e^{-\xi^2/4t}.
\end{aligned}$$

$$\boxed{\widehat{xe^{-tx^2}} = \frac{-i\xi}{(2t)^{3/2}} e^{-\xi^2/4t}.}$$

2. Use the Fourier transform to show that the solution of the inhomogeneous heat equation with zero initial data,

$$\begin{cases} u_t - ku_{xx} = f(x,t) & -\infty < x < \infty, t > 0 \\ u(x,0) = 0 & -\infty < x < \infty \end{cases}$$

is given by

$$u(x,t) = \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4k(t-s)} f(y,s)\, dy\, ds.$$

Answer: We take the Fourier transform with respect to the spacial variable only.

$$\widehat{u}_t = k\widehat{u_{xx}} + \widehat{f}(x,t)$$
$$\implies \widehat{u}_t + k\xi^2\widehat{u} = \widehat{f}(x,t).$$

We solve this first-order ODE using the integrating factor $e^{k\xi^2 t}$. Our solution is given by

$$\widehat{u}(\xi,t) = e^{-k\xi^2 t} \int_0^t e^{k\xi^2 s} \widehat{f}(\xi,s)\, ds + ce^{-k\xi^2 t}.$$

Now, $u(x,0) = 0 \implies \widehat{u}(\xi,0) = 0$. Therefore,

$$\widehat{u}(\xi,t) = e^{-k\xi^2 t} \int_0^t e^{k\xi^2 s} \widehat{f}(\xi,s)\, ds.$$

Using the fact that $u = \check{\widehat{u}}$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{u}(\xi,t)\, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \left[e^{-k\xi^2 t} \int_0^t e^{k\xi^2 s} \widehat{f}(\xi,s)\, ds \right] d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{ix\cdot\xi} \left[e^{-k\xi^2 t} \int_0^t e^{k\xi^2 s} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi} f(y,s)\, dy \right] ds \right] d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} f(y,s) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(y-x)\xi} e^{-k\xi^2(t-s)}\, d\xi \right] dy\, ds \end{aligned}$$

Notice that the inner term in brackets is just the Fourier transform $\widehat{g}(y-x)$ where $g(\xi) = e^{-k\xi^2(t-s)}$. From lecture, we know that for $\xi \in \mathbb{R}$,

$$g(\xi) = e^{-k(t-s)\xi^2} \implies \widehat{g}(x) = \frac{1}{(2k(t-s))^{1/2}} e^{-x^2/4k(t-s)}.$$

Using this fact, we can conclude that for $y, x \in \mathbb{R}$,

$$\widehat{g}(y-x) = \frac{1}{\sqrt{2k(t-s)}} e^{-(y-x)^2/4k(t-s)}.$$

Therefore,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} f(y,s) \widehat{g}(y-x)\, dy\, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} f(y,s) \frac{1}{\sqrt{2k(t-s)}} e^{-(y-x|)^2/4k(t-s)}\, dy\, ds \\ &= \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4k(t-s)} f(y,s)\, dy\, ds. \end{aligned}$$

3. Use the Fourier transform to solve

$$\begin{cases} u_t - tu_{xx} = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) & -\infty < x < \infty \end{cases}$$

Answer:

$$\begin{aligned} u_t - tu_{xx} = 0 &\implies \widehat{u}_t - t\widehat{u_{xx}} = 0 \\ &\implies \widehat{u}_t - t(i\xi)^2\widehat{u} = 0 \\ &\implies \widehat{u}_t + t\xi^2\widehat{u}. \end{aligned}$$

Solving this first-order ODE, we have

$$\widehat{u}(\xi,t) = Ce^{-t^2\xi^2/2}.$$

The initial condition $u(x,0) = \phi(x) \implies \widehat{u}(\xi,0) = \widehat{\phi}(\xi)$. Therefore,

$$\widehat{u}(\xi,t) = \widehat{\phi}(\xi)e^{-t^2\xi^2/2}.$$

Next, recall that

$$\widehat{f * g} = (2\pi)^{1/2}\widehat{f}(\xi)\widehat{g}(\xi).$$

Therefore,

$$\mathcal{F}^{-1}(\widehat{f}\widehat{g}) = \frac{1}{\sqrt{2\pi}} f * g.$$

Let $\widehat{f} = \widehat{\phi}$ and $\widehat{g}(\xi) = e^{-t^2\xi^2/2}$. We need to compute $g(x)$. Recall that

$$v(x) = e^{-\epsilon x^2} \implies \widehat{v}(\xi) = \frac{1}{\sqrt{2\epsilon}} e^{-\xi^2/4\epsilon}.$$

Therefore,

$$\widehat{g}(\xi) = e^{-t^2\xi^2/2} = \frac{1}{t}\left(\frac{1}{\sqrt{2(\frac{1}{2t^2})}}\right) e^{-\xi^2/4(\frac{1}{2t^2})} \implies g(x) = \frac{1}{t}e^{-x^2/2t^2}.$$

Therefore,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} f * g \\ &= \frac{1}{\sqrt{2\pi}} \phi * \frac{1}{t} e^{-x^2/2t^2}. \end{aligned}$$

Therefore,

$$\boxed{u(x,t) = \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2t^2} \phi(y)\, dy.}$$

4. (a) Consider the heat equation on a half-line with Dirichlet boundary conditions

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < \infty, t > 0 \\ u(x,0) = \phi(x) & 0 < x < \infty \\ u(0,t) = 0 & t > 0. \end{cases}$$

Solve for $u(x,t)$.

Answer: Let

$$\phi_{odd}(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases}$$

Consider the initial-value problem

$$\begin{cases} v_t - kv_{xx} = 0 & -\infty < x < \infty, t > 0 \\ v(x,0) = \phi_{odd}(x). \end{cases}$$

We know the solution of this initial value problem is given by

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi_{odd}(y)\, dy.$$

Using the fact that ϕ_{odd} is an odd function and $e^{-y^2/4kt}$ is an even function, we see that

$$v(0,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-y^2/4kt} \phi_{odd}(y)\, dy = 0.$$

Therefore, letting $u(x,t) = v(x,t)$ for $x \geq 0$, we see that $u(x,t)$ is a solution of the heat equation on the half-line with Dirichlet boundary conditions on the half-line. We conclude that

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi_{odd}(y)\, dy.}$$

(b) Consider the heat equation on a half-line with Robin boundary conditions

$$(*) \begin{cases} u_t - ku_{xx} = 0 & 0 < x < \infty, t > 0 \\ u(x,0) = \phi(x) & 0 < x < \infty \\ u_x(0,t) - hu(x,t) = 0 & t > 0. \end{cases}$$

Solve this initial value problem as follows. Assuming u is the solution of (*), introduce a new function v such that $v(x,t) = u_x(x,t) - hu(x,t)$.

i. Determine the initial/boundary value problem that v satisfies.

Answer:

$$\boxed{\begin{cases} v_t - kv_{xx} = 0 & 0 < x < \infty, t > 0 \\ v(x,0) = \phi'(x) - h\phi(x) & \\ v(0,t) = 0. & \end{cases}}$$

ii. Solve for u in terms of v.

Answer:

$$\begin{aligned} u_x - hu = v &\implies (e^{-hx}u)_x = e^{-hx}v \\ &\implies e^{-hx}u = \int_a^x e^{-hy}v(y,t)\,dy + C. \end{aligned}$$

Therefore,

$$\boxed{u(x,t) = e^{hx}\int_a^x e^{-hy}v(y,t)\,dy + e^{hx}C}$$

for any a and C. In order that the initial condition is satisfied, choose a and C such that $u(x,0) = \phi(x)$. (There is not a unique solution to this ODE.) For example, if we assume that ϕ is bounded, then let $C = 0$, $a = \infty$. In this case, we see that if $a = \infty$ (for $h > 0$), then

$$\begin{aligned} u(x,0) &= e^{hx}\int_\infty^x e^{-hy}[\phi' - h\phi]\,dy \\ &= e^{hx}\int_\infty^x e^{-hy}[h\phi - h\phi]\,dy + e^{hx}e^{-hy}\phi(y)|_{y\to\infty}^{y=x} \\ &= \phi(x). \end{aligned}$$

Similarly, if $h < 0$, then let $a = -\infty$.

5. Consider the initial/boundary-value problem

$$(**)\qquad \begin{cases} u_t - ku_{xx} = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) & 0 < x < l \\ u(0,t) = 0 = u(l,t) & t > 0. \end{cases}$$

Let $\phi_{ext}(x)$ be the extension of ϕ to all of $\mathbb{R}$ such that ϕ_{ext} is odd with respect to $x = 0$ and ϕ_{ext} is $2l$-periodic. That is,

$$\phi_{ext}(x) = \begin{cases} \phi(x) & 0 < x < l \\ -\phi(-x) & -l < x < 0 \end{cases}$$

and ϕ is $2l$-periodic.

(a) Consider the initial-value problem

$$\begin{cases} v_t - kv_{xx} = 0 & -\infty < x < \infty, t > 0 \\ v(x,0) = \phi_{ext}(x) & -\infty < x < \infty. \end{cases}$$

Write the solution formula for v. Show that if $u(x,t)$ is defined to be $v(x,t)$ for $0 \le x \le l$, then u will satisfy (**).

Answer:

$$\boxed{v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(y-x)^2/4kt}\phi_{ext}(y)\,dy.}$$

We know v satisfies the heat equation on $\mathbb{R}$, and, therefore, u will satisfy the heat equation on $\{(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+\}$. Also, $v(x,0) = \phi_{ext}(x)$ implies $v(x,0) = \phi(x)$ for $x > 0$, and, therefore, $u(x,0) = \phi(x)$ for $x > 0$. Therefore, the only thing we must check is that $u(0,t) = 0 = u(l,t)$. First, by definition of v, we see that

$$u(0,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-y^2/4kt} \phi_{ext}(y)\, dy.$$

Now using the fact that $\phi_{ext}(y)$ is odd with respect to $y = 0$ and $e^{-y^2/4kt}$ is even with respect to $y = 0$, we conclude that their product is odd, and, thus, $u(0,t) = 0$.

Second, we see that

$$u(l,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-l)^2/4kt} \phi_{ext}(y)\, dy.$$

By the change of variables $\widetilde{y} = y - l$, we can rewrite

$$u(l,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\widetilde{y}^2/4kt} \phi_{ext}(\widetilde{y} + l)\, d\widetilde{y}.$$

We note that $\phi_{ext}(y)$ is odd with respect to $y = l$. Therefore, $\phi_{ext}(y + l)$ (which is the function $\phi_{ext}(y)$ shifted to the left by l units) is odd with respect to $y = 0$. Also, as stated above, $e^{-y^2/4kt}$ is even, and, therefore, the product of these two functions is odd, which implies that $u(l,t) = 0$.

(b) Assume that

$$\phi(x) = \sum_{n=1} A_n \sin\left(\frac{n\pi}{l} x\right) \text{ for } 0 \le x \le l,$$

where

$$A_n = \frac{\langle \phi, \sin\left(\frac{n\pi}{l} x\right)\rangle}{\langle \sin\left(\frac{n\pi}{l} x\right), \sin\left(\frac{n\pi}{l} x\right)\rangle}.$$

(That is, assume that the Fourier sine series for ϕ converges to ϕ.) Note that for ϕ_{ext} defined above,

$$\phi_{ext}(x) = \sum_{n=1} A_n \sin\left(\frac{n\pi}{l} x\right) \text{ for } -\infty < x < \infty.$$

Using the solution formula found in part (a), show that

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l} x\right) e^{-kn^2\pi^2 t/l^2}$$

with A_n defined above. (Consequently if $u(x,t) = v(x,t)$ for $0 \le x \le l$, then u has this form. In particular, we have justified the separation of variables technique.)

Answer: By the formula in part (a), we know that

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4kt} \phi_{ext}(y)\, dy.$$

Using the facts stated above, we have that

$$\phi_{ext}(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right),$$

where A_n is defined above. Plugging this into the formula for v, we have that

$$\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4kt} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}y\right)\, dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} A_n \left(\frac{e^{in\pi y/l} - e^{-in\pi y/l}}{2i}\right) e^{-(y-x)^2/4kt}\, dy \\
&= \frac{1}{2i\sqrt{4\pi kt}} \sum_{n=1}^{\infty} \left[\int_{-\infty}^{\infty} e^{in\pi y/l} e^{-(y-x)^2/4kt}\, dy + \int_{-\infty}^{\infty} e^{-in\pi y/l} e^{-(y-x)^2/4kt}\, dy\right].
\end{aligned}$$

We look at the first term on the RHS above. Consider

$$\int_{-\infty}^{\infty} e^{-(y-x)^2/4kt} e^{in\pi y/l}\, dy.$$

Consider the exponent $-(y-x)^2/4kt + in\pi y/l$. We will complete the square.

$$\begin{aligned}
-\frac{(y-x)^2}{4kt} + \frac{in\pi}{l}y &= -\frac{(y-x)^2}{4kt} + i\frac{n\pi}{l}(y-x) + i\frac{n\pi}{l}x \\
&= -\frac{1}{4kt}\left[(y-x)^2 - i\frac{4ktn\pi}{l}(y-x) + \left(\frac{i4ktn\pi}{2l}\right)^2\right] \\
&\quad + \frac{1}{4kt}\left(\frac{i4ktn\pi}{2l}\right)^2 + i\frac{n\pi}{l}x \\
&= -\frac{1}{4kt}\left[(y-x) + \frac{i2ktn\pi}{l}\right]^2 - kt\left(\frac{n\pi}{l}\right)^2 + i\frac{n\pi}{l}x.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{in\pi y/l} e^{-(y-x)^2/4kt}\, dy &= \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}\left[(y-x)+i\frac{2ktn\pi}{l}\right]^2} e^{-kt\left(\frac{n\pi}{l}\right)^2} e^{i\frac{n\pi}{l}x}\, dy \\
&= e^{-kt\left(\frac{n\pi}{l}\right)^2} e^{i\frac{n\pi}{l}x} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(y-x)^2}\, dy.
\end{aligned}$$

Letting $z = (y-x)/\sqrt{4kt}$, we see that

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{in\pi y/l} e^{-(y-x)^2/4kt}\, dy &= e^{-kt\left(\frac{n\pi}{l}\right)^2} e^{i\frac{n\pi}{l}x} \sqrt{4kt} \int_{-\infty}^{\infty} e^{-z^2}\, dz \\
&= \sqrt{4\pi kt} e^{-kt\left(\frac{n\pi}{l}\right)^2} e^{i\frac{n\pi}{l}x}.
\end{aligned}$$

Doing a similar analysis for the term involving $e^{-in\pi x/l}$, we conclude that

$$\begin{aligned} v(x,t) &= \frac{1}{2i}\sum_{n=1}^{\infty} A_n \left(e^{i\frac{n\pi}{l}x} - e^{-i\frac{n\pi}{l}x}\right) e^{-kt\left(\frac{n\pi}{l}\right)^2} \\ &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) e^{-kt\left(\frac{n\pi}{l}\right)^2}. \end{aligned}$$