Math 220B - Summer 2003
Homework 2 Solutions

1. (a) Compute the Fourier transform of $xf$ in terms of $\hat{f}$.
   Answer:
   \[
   \hat{xf}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} xf(x) \, dx \\
   = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{-i} d\xi \left( e^{-ix\xi} f(x) \right) dx \\
   = \frac{1}{-i} d\xi \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \, dx \right) \\
   = i \frac{d}{d\xi} \hat{f}(\xi).
   \]
   \[
   \hat{xf}(\xi) = i \frac{d}{d\xi} \hat{f}(\xi).
   \]
   (b) Compute the Fourier transform of $xe^{-tx^2}$.
   Answer: From part (a),
   \[
   \hat{xe^{-tx^2}} = i \frac{d}{d\xi} \left( e^{-tx^2} \right).
   \]
   Recall $f(x) = e^{-cx^2}$ implies $\hat{f}(\xi) = \frac{1}{\sqrt{2c}} e^{-\xi^2/4c}$. Therefore, $e^{-tx^2} = \frac{1}{\sqrt{2t}} e^{-\xi^2/4t}$.
   Therefore,
   \[
   \hat{xe^{-tx^2}} = i \frac{d}{d\xi} \left( \frac{1}{\sqrt{2t}} e^{-\xi^2/4t} \right) \\
   = \frac{i}{\sqrt{2t}} \cdot \frac{-\xi}{2t} e^{-\xi^2/4t} \\
   = \frac{-i\xi}{(2t)^{3/2}} e^{-\xi^2/4t}.
   \]
   \[
   \hat{xe^{-tx^2}} = \frac{-i\xi}{(2t)^{3/2}} e^{-\xi^2/4t}.
   \]

2. Use the Fourier transform to show that the solution of the inhomogeneous heat equation with zero initial data,
   \[
   \begin{cases}
   u_t - ku_{xx} = f(x,t) & -\infty < x < \infty, t > 0 \\
   u(x,0) = 0 & -\infty < x < \infty
   \end{cases}
   \]
is given by
\[ u(x, t) = \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4k(t-s)} f(y, s) \, dy \, ds. \]

**Answer:** We take the Fourier transform with respect to the spacial variable only.

\[ \hat{u}_t = k \hat{u}_{xx} + \hat{f}(x, t) \]

\[ \implies \hat{u}_t + k\xi^2 \hat{u} = \hat{f}(x, t). \]

We solve this first-order ODE using the integrating factor $e^{k\xi^2 t}$. Our solution is given by

\[ \hat{u}(\xi, t) = e^{-k\xi^2 t} \int_0^t e^{k\xi^2 s} \hat{f}(\xi, s) \, ds + ce^{-k\xi^2 t}. \]

Now, $u(x, 0) = 0 \implies \hat{u}(\xi, 0) = 0$. Therefore,

\[ \hat{u}(\xi, t) = e^{-k\xi^2 t} \int_0^t e^{k\xi^2 s} \hat{f}(\xi, s) \, ds. \]

Using the fact that $u = \hat{\check{u}}$, we have

\[ u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \hat{u}(\xi, t) \, d\xi \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \left[ e^{-k\xi^2 t} \int_0^t e^{k\xi^2 s} \hat{f}(\xi, s) \, ds \right] \, d\xi \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \left[ e^{-k\xi^2 t} \int_0^t e^{k\xi^2 s} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\xi} f(y, s) \, dy \right] \, ds \right] \, d\xi \]

Notice that the inner term in brackets is just the Fourier transform $\hat{g}(y - x)$ where $g(\xi) = e^{-k\xi^2(t-s)}$. From lecture, we know that for $\xi \in \mathbb{R}$,

\[ g(\xi) = e^{-k(t-s)\xi^2} \implies \hat{g}(x) = \frac{1}{(2k(t-s))^{1/2}} e^{-x^2/4k(t-s)}. \]

Using this fact, we can conclude that for $y, x \in \mathbb{R}$,

\[ \hat{g}(y - x) = \frac{1}{\sqrt{2k(t-s)}} e^{-(y-x)^2/4k(t-s)}. \]

Therefore,

\[ u(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} f(y, s) \hat{g}(y - x) \, dy \, ds \]

\[ = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} f(y, s) \frac{1}{\sqrt{2k(t-s)}} e^{-(y-x)^2/4k(t-s)} \, dy \, ds \]

\[ = \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4k(t-s)} f(y, s) \, dy \, ds. \]
3. Use the Fourier transform to solve

\[
\begin{cases}
  u_t - tu_{xx} = 0 & -\infty < x < \infty, t > 0 \\
  u(x, 0) = \phi(x) & -\infty < x < \infty
\end{cases}
\]

Answer:

\[
\begin{align*}
  u_t - tu_{xx} = 0 & \implies \hat{u}_t - t\hat{u}_{xx} = 0 \\
     & \implies \hat{u}_t - t(i\xi)^2\hat{u} = 0 \\
     & \implies \hat{u}_t + t\xi^2\hat{u}.
\end{align*}
\]

Solving this first-order ODE, we have

\[
\hat{u}(\xi, t) = Ce^{-t\xi^2/2}.
\]

The initial condition \( u(x, 0) = \phi(x) \implies \hat{u}(\xi, 0) = \hat{\phi}(\xi) \). Therefore,

\[
\hat{u}(\xi, t) = \hat{\phi}(\xi)e^{-t\xi^2/2}.
\]

Next, recall that

\[
\tilde{f} \ast g = (2\pi)^{1/2}\hat{f}(\xi)\hat{g}(\xi).
\]

Therefore,

\[
\mathcal{F}^{-1}(\hat{f} \ast g) = \frac{1}{\sqrt{2\pi}}f \ast g.
\]

Let \( \hat{f} = \hat{\phi} \) and \( \hat{g}(\xi) = e^{-t^2\xi^2/2} \). We need to compute \( g(x) \). Recall that

\[
\tilde{v}(x) = e^{-\xi^2} \implies \hat{v}(\xi) = \frac{1}{\sqrt{2\xi}}e^{-\xi^2/4\xi}.
\]

Therefore,

\[
\hat{g}(\xi) = e^{-t^2\xi^2/2} = \frac{1}{t} \left( \frac{1}{\sqrt{2(\frac{1}{2t})}} \right) e^{-\xi^2/4(\frac{1}{2t})} \implies g(x) = \frac{1}{t}e^{-x^2/2t^2}.
\]

Therefore,

\[
u(x, t) = \frac{1}{\sqrt{2\pi}}f \ast g
\]

\[
= \frac{1}{\sqrt{2\pi}}\phi \ast \frac{1}{t}e^{-x^2/2t^2}.
\]

Therefore,

\[
u(x, t) = \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2t^2} \phi(y) \, dy.
\]
4. (a) Consider the heat equation on a half-line with Dirichlet boundary conditions

\[
\begin{aligned}
\begin{cases}
  u_t - ku_{xx} = 0 & 0 < x < \infty, t > 0 \\
  u(x, 0) = \phi(x) & 0 < x < \infty \\
  u(0, t) = 0 & t > 0.
\end{cases}
\end{aligned}
\]

Solve for \( u(x, t) \).

**Answer:** Let

\[
\phi_{\text{odd}}(x) = \begin{cases}
  \phi(x) & x > 0 \\
  -\phi(-x) & x < 0
\end{cases}
\]

Consider the initial-value problem

\[
\begin{aligned}
\begin{cases}
  v_t - kv_{xx} = 0 & -\infty < x < \infty, t > 0 \\
  v(x, 0) = \phi_{\text{odd}}(x).
\end{cases}
\end{aligned}
\]

We know the solution of this initial value problem is given by

\[
v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi_{\text{odd}}(y) \, dy.
\]

Using the fact that \( \phi_{\text{odd}} \) is an odd function and \( e^{-y^2/4kt} \) is an even function, we see that

\[
v(0, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-y^2/4kt} \phi_{\text{odd}}(y) \, dy = 0.
\]

Therefore, letting \( u(x, t) = v(x, t) \) for \( x \geq 0 \), we see that \( u(x, t) \) is a solution of the heat equation on the half-line with Dirichlet boundary conditions on the half-line. We conclude that

\[
\boxed{u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi_{\text{odd}}(y) \, dy.}
\]

(b) Consider the heat equation on a half-line with Robin boundary conditions

\[
\begin{aligned}
\begin{cases}
  u_t - ku_{xx} = 0 & 0 < x < \infty, t > 0 \\
  u(x, 0) = \phi(x) & 0 < x < \infty \\
  u_x(0, t) - hu(x, t) = 0 & t > 0.
\end{cases}
\end{aligned}
\]

Solve this initial value problem as follows. Assuming \( u \) is the solution of (**), introduce a new function \( v \) such that \( v(x, t) = u_x(x, t) - hu(x, t) \).

i. Determine the initial/boundary value problem that \( v \) satisfies.

**Answer:**

\[
\begin{aligned}
\begin{cases}
  v_t - kv_{xx} = 0 & 0 < x < \infty, t > 0 \\
  v(x, 0) = \phi'(x) - h\phi(x) \\
  v(0, t) = 0.
\end{cases}
\end{aligned}
\]
ii. Solve for \( u \) in terms of \( v \).

**Answer:**

\[
\begin{align*}
  u_x - hu &= v \quad \Rightarrow \quad (e^{-hx}u)_x = e^{-hx}v \\
  &\quad \Rightarrow e^{-hx}u = \int_a^x e^{-hy}v(y, t) \, dy + C.
\end{align*}
\]

Therefore,

\[
  \boxed{u(x, t) = e^{hx} \int_a^x e^{-hy}v(y, t) \, dy + e^{hx}C}
\]

for any \( a \) and \( C \). In order that the initial condition is satisfied, choose \( a \) and \( C \) such that \( u(x, 0) = \phi(x) \). (There is not a unique solution to this ODE.)

For example, if we assume that \( \phi \) is bounded, then let \( C = 0, \ a = \infty \). In this case, we see that if \( a = \infty \) (for \( h > 0 \)), then

\[
\begin{align*}
  u(x, 0) &= e^{hx} \int_a^x e^{-hy}[\phi' - h\phi] \, dy \\
  &= e^{hx} \int_a^x e^{-hy}[h\phi - h\phi] \, dy + e^{hx}e^{-hy}\phi(y)|_{y=\infty}^{y=x} \\
  &= \phi(x).
\end{align*}
\]

Similarly, if \( h < 0 \), then let \( a = -\infty \).

5. Consider the initial/boundary-value problem

\[
\begin{align*}
  (**) \quad \left\{ \begin{array}{ll}
  u_t - ku_{xx} = 0 & 0 < x < l, t > 0 \\
  u(x, 0) = \phi(x) & 0 < x < l \\
  u(0, t) = 0 = u(l, t) & t > 0.
\end{array} \right.
\end{align*}
\]

Let \( \phi_{ext}(x) \) be the extension of \( \phi \) to all of \( \mathbb{R} \) such that \( \phi_{ext} \) is odd with respect to \( x = 0 \) and \( \phi_{ext} \) is 2\( l \)-periodic. That is,

\[
\phi_{ext}(x) = \left\{ \begin{array}{ll}
  \phi(x) & 0 < x < l \\
  -\phi(-x) & -l < x < 0
\end{array} \right.
\]

and \( \phi \) is 2\( l \)-periodic.

(a) Consider the initial-value problem

\[
\begin{align*}
  \left\{ \begin{array}{ll}
  v_t - kv_{xx} = 0 & -\infty < x < \infty, t > 0 \\
  v(x, 0) = \phi_{ext}(x) & -\infty < x < \infty.
\end{array} \right.
\end{align*}
\]

Write the solution formula for \( v \). Show that if \( u(x, t) \) is defined to be \( v(x, t) \) for \( 0 \leq x \leq l \), then \( u \) will satisfy (**).

**Answer:**

\[
v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{-(y-x)^2/4kt} \phi_{ext}(y) \, dy.
\]
We know $v$ satisfies the heat equation on $\mathbb{R}$, and, therefore, $u$ will satisfy the heat equation on $\{(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+\}$. Also, $v(x, 0) = \phi_{\text{ext}}(x)$ implies $v(x, 0) = \phi(x)$ for $x > 0$, and, therefore, $u(x, 0) = \phi(x)$ for $x > 0$. Therefore, the only thing we must check is that $u(0, t) = 0 = u(l, t)$. First, by definition of $v$, we see that

$$u(0, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-y^2/4kt} \phi_{\text{ext}}(y) \, dy.$$ 

Now using the fact that $\phi_{\text{ext}}(y)$ is odd with respect to $y = 0$ and $e^{-y^2/4kt}$ is even with respect to $y = 0$, we conclude that their product is odd, and, thus, $u(0, t) = 0$.

Second, we see that

$$u(l, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-l)^2/4kt} \phi_{\text{ext}}(y) \, dy.$$ 

By the change of variables $\tilde{y} = y - l$, we can rewrite

$$u(l, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\tilde{y}^2/4kt} \phi_{\text{ext}}(\tilde{y} + l) \, d\tilde{y}.$$ 

We note that $\phi_{\text{ext}}(y)$ is odd with respect to $y = l$. Therefore, $\phi_{\text{ext}}(y + l)$ (which is the function $\phi_{\text{ext}}(y)$ shifted to the left by $l$ units) is odd with respect to $y = 0$. Also, as stated above, $e^{-y^2/4kt}$ is even, and, therefore, the product of these two functions is odd, which implies that $u(l, t) = 0$.

(b) Assume that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{l} x \right) \quad \text{for} \quad 0 \leq x \leq l,$$

where

$$A_n = \frac{\langle \phi, \sin \left( \frac{n\pi}{l} x \right) \rangle}{\langle \sin \left( \frac{n\pi}{l} x \right), \sin \left( \frac{n\pi}{l} x \right) \rangle}.$$ 

(That is, assume that the Fourier sine series for $\phi$ converges to $\phi$.) Note that for $\phi_{\text{ext}}$ defined above,

$$\phi_{\text{ext}}(x) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{l} x \right) \quad \text{for} \quad -\infty < x < \infty.$$ 

Using the solution formula found in part (a), show that

$$v(x, t) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{l} x \right) e^{-kn^2\pi^2t/l^2}$$

with $A_n$ defined above. (Consequently if $u(x, t) = v(x, t)$ for $0 \leq x \leq l$, then $u$ has this form. In particular, we have justified the separation of variables technique.)

**Answer:** By the formula in part (a), we know that

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4kt} \phi_{\text{ext}}(y) \, dy.$$
Using the facts stated above, we have that

$$\phi_{\text{ext}}(y) = \sum_{n=1}^{\infty} A_n \sin \left(\frac{n\pi}{l}x\right),$$

where $A_n$ is defined above. Plugging this into the formula for $v$, we have that

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4kt} \sum_{n=1}^{\infty} A_n \sin \left(\frac{n\pi}{l} y\right) dy \tag{1}$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} A_n \left(\frac{e^{in\pi y/l} - e^{-in\pi y/l}}{2i}\right) e^{-(y-x)^2/4kt} dy \tag{2}$$

$$= \frac{1}{2i\sqrt{4\pi kt}} \sum_{n=1}^{\infty} \left[ \int_{-\infty}^{\infty} e^{in\pi y/l} e^{-(y-x)^2/4kt} dy + \int_{-\infty}^{\infty} e^{-in\pi y/l} e^{-(y-x)^2/4kt} dy \right]. \tag{3}$$

We look at the first term on the RHS above. Consider

$$\int_{-\infty}^{\infty} e^{-(y-x)^2/4kt} e^{in\pi y/l} dy. \tag{4}$$

Consider the exponent $-(y - x)^2/4kt + in\pi y/l$. We will complete the square.

$$-(y - x)^2 + in\pi y/l = -(y - x)^2 + \frac{n\pi}{l} (y - x) + \frac{n\pi}{l} x$$

$$= -\frac{1}{4kt} \left[ (y - x)^2 - i\frac{4ktn\pi}{l} (y - x) + \left(\frac{4ktn\pi}{2l}\right)^2 \right]$$

$$+ \frac{1}{4kt} \left(\frac{i4ktn\pi}{2l}\right)^2 + i\frac{n\pi}{l} x$$

$$= -\frac{1}{4kt} \left[ (y - x) + \frac{i2ktn\pi}{l}\right]^2 - kt \left(\frac{n\pi}{l}\right)^2 + i\frac{n\pi}{l} x. \tag{5}$$

Therefore,

$$\int_{-\infty}^{\infty} e^{in\pi y/l} e^{-(y-x)^2/4kt} dy = \int_{-\infty}^{\infty} e^{-\frac{1}{4kt} \left[(y-x) + i\frac{2ktn\pi}{l}\right]^2} e^{-kt \left(\frac{n\pi}{l}\right)^2} e^{i\frac{n\pi}{l} x} dy$$

$$= e^{-kt \left(\frac{n\pi}{l}\right)^2} e^{i\frac{n\pi}{l} x} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt} (y-x)^2} dy. \tag{6}$$

Letting $z = (y - x)/\sqrt{4kt}$, we see that

$$\int_{-\infty}^{\infty} e^{in\pi y/l} e^{-(y-x)^2/4kt} dy = e^{-kt \left(\frac{n\pi}{l}\right)^2} e^{i\frac{n\pi}{l} x} \sqrt{4\pi kt} \int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= \sqrt{4\pi kt} e^{-kt \left(\frac{n\pi}{l}\right)^2} e^{i\frac{n\pi}{l} x}. \tag{7}$$
Doing a similar analysis for the term involving $e^{-in\pi x/l}$, we conclude that

$$v(x, t) = \frac{1}{2i} \sum_{n=1}^{\infty} A_n \left( e^{in\pi x} - e^{-in\pi x} \right) e^{-kt(n\pi)^2}$$

$$= \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{l} x \right) e^{-kt(n\pi)^2}.$$