Math 220B - Summer 2003
Homework 3 Solutions

1. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of continuous, nonnegative functions such that

$$f_n(x) \to \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

as $n \to +\infty$. In addition, assume

$$\int_{-\infty}^{\infty} f_n(x)\, dx = 1$$

for all n, and there exists a closed, bounded subset $K \subset \mathbb{R}$ such that $f_n(x) \equiv 0$ for $x \notin K$. (That is, $\operatorname{supp}(f_n) \subset K$ for all n.)

Show that

$$f_n \rightharpoonup \delta_0$$

as $n \to +\infty$ in the sense of distributions. (That is, prove weak convergence.)

Answer: *Note:* The above statement does *not* hold without a stronger assumption on the sequence $\{f_n\}$. Below, I will provide a proof under the following stronger assumption:

- The sequence f_n converges to the zero function *uniformly* (not just pointwise) on any compact set $\Omega \subset \mathbb{R}$ such that $0 \notin \Omega$.

(*Remark:* The stated problem can also be proven under different assumptions on $\{f_n\}$. For example, as long as f_n is uniformly bounded on any compact set not containing the origin, then the result holds. Under this assumption, the stated problem can be proven using Lebesgue's Dominated Convergence Theorem.)

Define the distribution $F_{f_n} : \mathcal{D} \to \mathbb{R}$ such that

$$(F_{f_n}, \phi) = \int_{-\infty}^{\infty} f_n(x)\phi(x)\, dx.$$

To show that $f_n \rightharpoonup \delta_0$ in the sense of distributions we need to show that

$$(F_{f_n}, \phi) \to (\delta_0, \phi) \qquad \forall \phi \in \mathcal{D}.$$

Using the definition of F_{f_n} and δ_0 that means we need to show that

$$\left| \int_{-\infty}^{\infty} f_n(x)\phi(x)\, dx - \phi(0) \right| \to 0 \text{ as } n \to +\infty.$$

In particular, we need to show that for all $\epsilon > 0$ there exists $N > 0$ such that

$$\left| \int_{-\infty}^{\infty} f_n(x)\phi(x)\, dx - \phi(0) \right| < \epsilon$$

if $n \geq N$.

By assumption $\int_{-\infty}^{\infty} f_n(x)\,dx = 1$ for all n. Therefore,

$$\phi(0) = \int_{-\infty}^{\infty} f_n(x)\phi(0)\,dx.$$

Therefore, we need to consider

$$\left|\int_{-\infty}^{\infty} f_n(x)[\phi(x) - \phi(0)]\,dx\right|.$$

We divide $\mathbb{R}$ into two pieces: $B(0,\delta)$ and $\mathbb{R}\backslash B(0,\delta)$, where we will determine δ below. Now

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f_n(x)[\phi(x) - \phi(0)]\,dx\right| &\leq \left|\int_{B(0,\delta)} f_n(x)[\phi(x) - \phi(0)]\,dx\right| \\ &\quad + \left|\int_{\mathbb{R}\backslash B(0,\delta)} f_n(x)[\phi(x) - \phi(0)]\,dx\right| \\ &\equiv I + J.\end{aligned}$$

First, we look at term I. Using the assumption that ϕ is continuous and f_n is nonnegative, we see that

$$\begin{aligned}\left|\int_{B(0,\delta)} f_n(x)[\phi(x) - \phi(0)]\,dx\right| &\leq |\phi(x) - \phi(0)|_{L^\infty(B(0,\delta))} \int_{B(0,\delta)} |f_n(x)|\,dx \\ &= |\phi(x) - \phi(0)|_{L^\infty(B(0,\delta))} \int_{B(0,\delta)} f_n(x)\,dx \\ &\leq |\phi(x) - \phi(0)|_{L^\infty(B(0,\delta))} \int_{\mathbb{R}} f_n(x)\,dx \\ &= |\phi(x) - \phi(0)|_{L^\infty(B(0,\delta))} \\ &< \frac{\epsilon}{2},\end{aligned}$$

by taking δ sufficiently small.

Next for term J, using the fact that the support of f_n is contained within K for all n, and ϕ is bounded, we see that

$$\begin{aligned}\left|\int_{\mathbb{R}\backslash B(0,\delta)} f_n(x)[\phi(x) - \phi(0)]\,dx\right| &\leq C \int_{K\backslash B(0,\delta)} |f_n(x)|\,dx \\ &\leq C|f_n(x)|_{L^\infty(K\backslash B(0,\delta))}.\end{aligned}$$

Now if we assume that $f_n \to 0$ uniformly on any compact set Ω not containing the origin, then, in particular, there exists N such that

$$C|f_n(x)|_{L^\infty(K\backslash B(0,\delta))} < \frac{\epsilon}{2}$$

for $n \geq N$.

2. (a) We define the Fourier transform of a distribution as follows. Let $F : \mathcal{D} \to \mathbb{R}$ be a distribution. Then $\widehat{F}$ is defined as the distribution such that $(\widehat{F}, \phi) = (F, \widehat{\phi})$ for all $\phi \in \mathcal{D}$. Using this definition, compute the Fourier transform of the delta function.

Answer: By the definition above, we have

$$\begin{aligned}
(\widehat{\delta_0}, \phi) &\equiv (\delta_0, \widehat{\phi}) \\
&= \widehat{\phi}(0) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\cdot 0} \phi(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x)\, dx.
\end{aligned}$$

Therefore,

$$\boxed{(\widehat{\delta_0}, \phi) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \phi(x)\, dx.}$$

We say the Fourier transform of the delta function is $1/\sqrt{2\pi}$.

(b) Use your answer to part (a) to solve

$$y'(x) - ay = \delta_0 \qquad a \in \mathbb{R}, a \neq 0.$$

Answer: Taking the Fourier transform of the equation, we have

$$\begin{aligned}
\widehat{y'} - a\widehat{y} = \widehat{\delta_0} &\implies (i\xi)\widehat{y} - a\widehat{y} = \frac{1}{\sqrt{2\pi}} \\
&\implies \widehat{y} = \frac{1}{[i\xi - a]\sqrt{2\pi}}.
\end{aligned}$$

In particular, we have

$$\widehat{y} = -\frac{i\xi + a}{[\xi^2 + a^2]\sqrt{2\pi}}$$

Using a Fourier transform table, we know that

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{a}{\xi^2 + a^2} \implies f(x) = e^{-a|x|}.$$

Therefore,

$$-\mathcal{F}(e^{-a|x|})(\xi) = -\frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + \xi^2}.$$

In addition, we know that

$$\begin{aligned}
-\frac{i\xi}{[\xi^2 + a^2]\sqrt{2\pi}} &= \frac{i\xi}{a} \cdot \left[-\frac{1}{\sqrt{2\pi}} \frac{a}{a^2 + \xi^2} \right] \\
&= \frac{i\xi}{2a} \cdot \mathcal{F}(-e^{-a|x|})(\xi) \\
&= -\frac{1}{2a} \mathcal{F}(\partial_x(e^{-a|x|}))(\xi)
\end{aligned}$$

Now computing the distributional derivative of $e^{-a|x|}$, we see that

$$(F_{e^{-a|x|}}, \phi) = \int_{-\infty}^{\infty} e^{-a|x|}\phi(x)\,dx$$

implies

$$\begin{aligned}
(F'_{e^{-a|x|}}, \phi) &= -\int_{-\infty}^{\infty} e^{-a|x|}\phi'(x)\,dx \\
&= -\int_{0}^{\infty} e^{-ax}\phi'(x)\,dx - \int_{-\infty}^{0} e^{ax}\phi'(x)\,dx \\
&= -a\int_{0}^{\infty} e^{-ax}\phi(x)\,dx - e^{-ax}\phi(x)\big|_{x=0}^{x\to+\infty} \\
&\qquad + a\int_{-\infty}^{0} e^{ax}\phi(x)\,dx - e^{ax}\phi(x)\big|_{x\to-\infty}^{x=0} \\
&= -a\int_{-\infty}^{\infty} \operatorname{sgn}(x)e^{-a|x|}\phi(x)\,dx
\end{aligned}$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Therefore,

$$\begin{aligned}
-\frac{i\xi}{[\xi^2 + a^2]\sqrt{2\pi}} &= \frac{1}{2a}\mathcal{F}(a\operatorname{sgn}(x)e^{-a|x|})(\xi) \\
&= \frac{1}{2}\mathcal{F}(\operatorname{sgn}(x)e^{-a|x|})(\xi).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\widehat{y}(\xi) &= -\frac{1}{\sqrt{2\pi}}\frac{a}{a^2 + \xi^2} - \frac{i\xi}{[\xi^2 + a^2]\sqrt{2\pi}} \\
&= -\frac{1}{2}\mathcal{F}(e^{-a|x|})(\xi) + \frac{1}{2}\mathcal{F}(\operatorname{sgn}(x)e^{-a|x|})(\xi) \\
&= -\mathcal{F}(H(-x)e^{-a|x|})(\xi).
\end{aligned}$$

We conclude that

$$\boxed{y(x) = -H(-x)e^{-a|x|} = \begin{cases} 0 & x > 0 \\ -e^{ax} & x < 0 \end{cases}}$$

3. Define

$$u(x) = \begin{cases} 0 & x < 0 \\ \sin x & x \geq 0 \end{cases}$$

Show that $u'' + u = \delta_0$ in the sense of distributions, where δ denotes the delta function.

Answer: Let F_u be the distribution defined such that

$$(F_u, \phi) = \int_{-\infty}^{\infty} u(x)\phi(x)\,dx$$

for $\phi \in \mathcal{D}$. To show that $u'' + u = \delta_0$ in the sense of distributions, we need to show that

$$(F_u'' + F_u, \phi) = (\delta_0, \phi) = \phi(0).$$

By definition,

$$(F_u'', \phi) = (F_u, \phi'').$$

Therefore, we just need to show that

$$(F_u, \phi'') + (F_u, \phi) = \phi(0).$$

Now

$$\begin{aligned}
(F_u, \phi'') + (F_u, \phi) &= \int_{-\infty}^{\infty} u(x)\phi''(x)\,dx + \int_{-\infty}^{\infty} u(x)\phi(x)\,dx \\
&= \int_0^{\infty} \sin(x)\phi''(x)\,dx + \int_0^{\infty} \sin(x)\phi(x)\,dx \\
&= -\int_0^{\infty} \cos(x)\phi'(x)\,dx + \sin(x)\phi(x)|_{x=0}^{x=\infty} + \int_0^{\infty} \sin(x)\phi(x)\,dx \\
&= -\int_0^{\infty} \sin(x)\phi(x)\,dx - \cos(x)\phi(x)|_{x=0}^{x=\infty} + \sin(x)\phi(x)|_{x=0}^{x=\infty} \\
&\qquad + \int_0^{\infty} \sin(x)\phi(x)\,dx \\
&= \cos(0)\phi(0) = \phi(0).
\end{aligned}$$

4. Solve

$$\begin{cases} u_t - ku_{xx} + u = 0 & 0 < x < 1, t > 0 \\ u(x,0) = \phi(x) \\ u_x(0,t) - u(0,t) = g(t) \\ u_x(1,t) = h(t). \end{cases} \tag{1}$$

Answer: Let

$$\mathcal{U}(x,t) = h(t)x + [h(t) - g(t)].$$

Then assuming u is a solution of the stated problem, for

$$v(x,t) \equiv u(x,t) - \mathcal{U}(x,t),$$

we see that v satisfies

$$\begin{cases} v_t - kv_{xx} + v = -[h'(t)x + h'(t) - g'(t) + h(t)x + h(t) - g(t)] \equiv F(x,t) \\ v(x,0) = \phi(x) - [h(0)x + (h(0) - g(0))] \equiv \Phi(x) \\ v_x(0,t) - v(0,t) = 0 \\ v_x(1,t) = 0. \end{cases} \tag{2}$$

Now look at the homogeneous equation

$$\begin{cases} v_t - kv_{xx} + v = 0 & 0 < x < 1 \\ v(x,0) = \Phi(x) \\ v_x(0,t) - v(0,t) = 0 \\ v_x(1,t) = 0. \end{cases}$$

We solve this using separation of variables. Look for a solution of the form $v(x,t) = X(x)T(t)$. Plugging this into the equation, we have

$$\frac{T'}{kT} = \frac{X'' - \frac{1}{k}X}{X} = -\lambda.$$

We need to solve the eigenvalue problem

$$\begin{cases} X'' = \left(\frac{1}{k} - \lambda\right) X & 0 < x < 1 \\ X'(0) - X(0) = 0 \\ X'(1) = 0 \end{cases}$$

Let $-\mu \equiv \frac{1}{k} - \lambda$. That is, we need to solve

$$(*) \begin{cases} X'' = -\mu X & 0 < x < 1 \\ X'(0) - X(0) = 0 \\ X'(1) = 0. \end{cases}$$

We look for positive eigenvalues $\mu = \beta^2$. In this case, we have

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

The boundary condition

$$X'(0) - X(0) = 0 \implies A = B\beta.$$

The boundary condition

$$X'(1) = 0 \implies -A\beta\sin(\beta) + B\beta\cos(\beta) = 0.$$

Combining these two equations, we have

$$\tan(\beta) = \frac{1}{\beta}.$$

Therefore, we have eigenfunctions

$$X_n(x) = \beta_n \cos(\beta_n x) + \sin(\beta_n x)$$

for β_n satisfying the implicit equation $\tan(\beta_n) = \frac{1}{\beta_n}$.

One can check that all eigenvalues of (*) are positive.

Now solving for λ in terms of μ, we see that

$$\lambda_n = \frac{1}{k} + \beta_n^2 \text{ where } \tan(\beta_n) = \frac{1}{\beta_n}.$$

Solving our equation for T_n, we have

$$T_n(t) = C_n e^{-k\lambda_n t}.$$

Therefore, the solution operator $S(t)$ for our homogeneous equation is defined such that

$$S(t)\Phi = \sum_{n=1}^{\infty} C_n X_n(x) e^{-k\lambda_n t}$$

where

$$X_n(x) = \beta_n \cos(\beta_n x) + \sin(\beta_n x)$$

for β_n a solution of $\tan(\beta) = \frac{1}{\beta}$ and

$$C_n = \frac{\langle X_n, \Phi \rangle}{\langle X_n, X_n \rangle}.$$

By Duhamel's principle, the solution of the inhomogeneous equation is

$$v(x,t) = S(t)\Phi + \int_0^t S(t-s)F(s)\, ds.$$

Therefore, the solution of (2) is given by

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} C_n X_n(x) e^{-k\lambda_n t} + \int_0^t \sum_{n=1}^{\infty} D_n(s) X_n(x) e^{-k\lambda_n t}}$$

where

$$\boxed{\begin{aligned}
&C_n = \frac{\langle X_n, \Phi \rangle}{\langle X_n, X_n \rangle} \\
&D_n(s) = \frac{\langle X_n, F(s) \rangle}{\langle X_n, X_n \rangle} \\
&X_n(x) = \beta_n \cos(\beta_n x) + \sin(\beta_n x) \\
&\beta_n \text{ satisfies } \tan(\beta) = \frac{1}{\beta} \\
&\lambda_n = \frac{1}{k} + \beta_n^2 \\
&\Phi(x) = \phi(x) - [h(0)x + (h(0) - g(0))] \\
&F(x,t) = -[h'(t)x + h'(t) - g'(t) + h(t)x + h(t) - g(t)].
\end{aligned}}$$

Using the fact that

$$u(x,t) = v(x,t) + \mathcal{U}(x,t),$$

we conclude that the solution of (1) is given by

$$\boxed{u(x,t) = v(x,t) + h(t)x + [h(t) - g(t)]}$$

for $v(x,t)$ defined above.

5. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Use energy methods to prove uniqueness of solutions to

$$\begin{cases} u_t - k\Delta u + u = f & x \in \Omega, t > 0 \\ u(x,0) = \phi(x) & \\ \frac{\partial u}{\partial \nu} + au = g & x \in \partial\Omega \end{cases}$$

for $a \geq 0$.

Answer: Suppose there are two solutions u and v. Let $w \equiv u - v$. Then w satisfies

$$\begin{cases} w_t - k\Delta w + w = 0 & x \in \Omega, t > 0 \\ w(x,0) = 0 & \\ \frac{\partial w}{\partial \nu} + aw = 0 & x \in \partial\Omega. \end{cases} \tag{3}$$

Multiplying the equation by w and integrating over Ω, we see that

$$\begin{aligned} 0 &= \int_\Omega w(w_t - k\Delta w + w)\,dx \\ &= \int_\Omega \left[\left(\frac{1}{2}w^2\right)_t + k|\nabla w|^2 + w^2\right] dx - k\int_{\partial\Omega} w\frac{\partial w}{\partial \nu}\,dS \\ &= \partial_t\left(\frac{1}{2}\int_\Omega w^2\,dx\right) + \int_\Omega [k|\nabla w|^2 + w^2]\,dx + ak\int_{\partial\Omega} w^2\,dS. \end{aligned}$$

Therefore, defining

$$E_w(t) = \frac{1}{2}\int_\Omega w^2, dx,$$

we see from above, that for w a solution of (3),

$$E_w'(t) = -\int_\Omega [k|\nabla w|^2 + w^2]\,dx - ak\int_{\partial\Omega} w^2\,dS \leq 0.$$

Using the fact that $E_w(0) = 0$ for w a solution of (3) and $E_w(t) \geq 0$ (by definition, we see that $E_w(t) \equiv 0$. Therefore, $w^2(x,t) \equiv 0$ which implies $w(x,t) \equiv 0$. We conclude that $u \equiv v$.