Math 220B - Summer 2003
Homework 4 Solutions

1. Let Ω be an open, bounded set in $\mathbb{R}^n$. Let $\phi \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$, and suppose $u \in C^2(\overline{\Omega} \times [0,\infty))$ is a solution of

$$(1) \quad \begin{cases} u_t(x,t) = k\Delta u(x,t) & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = g(x) & (x,t) \in \partial\Omega \times [0,\infty) \\ u(x,0) = \phi(x) & x \in \Omega. \end{cases}$$

Also suppose that $v \in C^2(\overline{\Omega})$ is a solution of

$$(2) \quad \begin{cases} \Delta v(x) = 0 & x \in \Omega \\ v(x) = g(x) & x \in \partial\Omega. \end{cases}$$

Prove $u(x,t) \to v(x)$ in $L^2(\Omega)$ as $t \to \infty$ as follows.

(a) For Ω an open, bounded set in $\mathbb{R}^n$, there exists a constant C (depending only on Ω) such that for every $f \in C^1(\overline{\Omega})$ with $f(x) = 0$ for all $x \in \partial\Omega$,

$$(*) \quad \|f\|_{L^2(\Omega)} \leq C\|\nabla f\|\|_{L^2(\Omega)}.$$

Prove this inequality for the case when $\Omega = (a,b) \subset \mathbb{R}$.

Answer: Let

$$F(x) = \int_a^x f(y)\,dy.$$

Then integrating by parts and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} ||f||^2_{L^2([a,b])} &= \int_a^b f^2(x)\,dx \\ &= \left| -\int_a^b F(x)f'(x)\,dx + F(x)f(x)|_{x=a}^{x=b} \right| \\ &\leq \int_a^b |F(x)f'(x)|\,dx \\ &\leq \left(\int_a^b |F(x)|^2\,dx\right)^{1/2} \left(\int_a^b |f'(x)|^2\,dx\right)^{1/2}. \end{aligned}$$

Now

$$\left(\int_a^b F^2(x)\,dx\right)^{1/2} = \left(\int_a^b \left(\int_a^x f(y)\,dy\right)^2 dx\right)^{1/2}$$
$$\leq \left(\int_a^b \left(\int_a^b |f(y)|\,dy\right)^2 dx\right)^{1/2}$$
$$\leq (b-a)^{1/2}\left(\int_a^b |f(y)|\,dy\right)$$
$$\leq (b-a)^{1/2}\left(\int_a^b 1^2\,dy\right)^{1/2}\left(\int_a^b |f(y)|^2\,dy\right)^{1/2}$$
$$\leq (b-a)\left(\int_a^b |f(y)|^2\,dy\right)^{1/2}.$$

Therefore,

$$\int_a^b |f(x)|^2\,dx \leq (b-a)\left(\int_a^b |f(x)|^2\,dx\right)^{1/2}\left(\int_a^b |f'(x)|^2\,dx\right)^{1/2}$$

Dividing both sides by $\left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$, we get the desired result.

(b) Using (*), prove that for u and v solutions of (1) and (2) respectively, $u(x,t) \to v(x)$ in $L^2(\Omega)$ as $t \to +\infty$. (*Hint: Let* $w(x,t) \equiv u(x,t) - v(x)$. *Consider the PDE that* w *solves. Show that* $||w(x,t)||^2_{L^2(\Omega)} \to 0$ *as* $t \to +\infty$.)

Answer: Let $w(x,t) = u(x,t) - v(x)$. Then we see that w is a solution of

$$\begin{cases} w_t - k\Delta w = 0 & (x,t) \in \Omega \times (0,\infty) \\ w(x,0) = \phi(x) - v(x) & x \in \Omega \\ w(x,t) = 0 & x \in \partial\Omega, t > 0 \end{cases}$$

We want to show that $||w(x,t)||^2_{L^2(\Omega)} \to 0$ as $t \to \infty$. Now

$$\frac{1}{2}||w(x,t)||^2_{L^2(\Omega)} = \frac{1}{2}\int_\Omega w(x,t)^2\,dx$$

is the Energy function associated with the heat equation. Let

$$E(t) = \frac{1}{2}\int_\Omega w(x,t)^2\,dx.$$

As we have seen,

$$\begin{aligned} E'(t) &= \int_\Omega w w_t \, dx \\ &= \int_\Omega w k \Delta w \, dx \\ &= -k \int_\Omega |\nabla w|^2 \, dx + k \int_{\partial\Omega} w \frac{\partial w}{\partial \nu} \, dS(x) \\ &= -k \int_\Omega |\nabla w|^2 \, dx. \end{aligned}$$

But, using part (a), we see that

$$\begin{aligned} E'(t) &= -k \int_\Omega |\nabla w|^2 \, dx \\ &\leq -C \int_\Omega |w|^2 \, dx \\ &= -CE(t). \end{aligned}$$

Therefore, we see that

$$E(t) \leq K e^{-Ct}.$$

Therefore, $E(t) \to 0$ as $t \to +\infty$. In particular, this means $||w(x,t)||^2_{L^2(\Omega)} \to 0$ as $t \to +\infty$, as claimed.

2. Let $B_n(0,a)$ be the unit ball in $\mathbb{R}^n$ centered at 0 with radius $a > 0$.

(a) Let $\alpha > 0$. Show that

$$\int_{B_n(0,a)} \frac{1}{|x|^\alpha} \, dx < \infty$$

if and only if $\alpha < n$. In particular, evaluate the integral for $n > \alpha > 0$.

Answer: First, consider the case when $\alpha \neq n$. We have

$$\begin{aligned} \int_{B_n(0,a)} \frac{1}{|x|^\alpha} \, dx &= \int_0^a \int_{\partial B_n(0,r)} \frac{1}{|x|^\alpha} \, dS(x) \, dr \quad = \int_0^a \int_{\partial B_n(0,r)} \frac{1}{r^\alpha} \, dS(x) \, dr \\ &= \int_0^a \frac{1}{r^\alpha} \int_{\partial B_n(0,r)} dS(x) \, dr \\ &= \int_0^a \frac{1}{r^\alpha} n\alpha(n) r^{n-1} \, dr \\ &= \int_0^a n\alpha(n) r^{n-\alpha-1} \, dr \\ &= \frac{n\alpha(n)}{n-\alpha} r^{n-\alpha} \Big|_{r \to 0}^{r=a} \end{aligned}$$

We see that this limit is finite if and only if $\alpha < n$. In particular, for $0 < \alpha < n$, we have

$$\boxed{\int_{B_n(0,a)} \frac{1}{|x|^\alpha}\,dx = \frac{n\alpha(n)}{n-\alpha}a^{n-\alpha}.}$$

Lastly, we consider the case when $\alpha = n$. In this case, we see that the integral becomes

$$\int_0^a n\alpha(n)r^{n-\alpha-1}\,dr = \int_0^a n\alpha(n)r^{-1}\,dr = n\alpha(n)\ln(r)\big|_{r\to 0}^{r=a} \to +\infty.$$

Therefore, the integral is not finite for $\alpha = n$.

(b) Give conditions on α for which

$$\int_{\mathbb{R}^n \setminus B_n(0,a)} \frac{1}{|x|^\alpha}\,dx < \infty.$$

Answer: Again, consider $\alpha \neq n$. As in part (a), we write

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_n(0,a)} \frac{1}{|x|^\alpha}\,dx &= \int_a^\infty \int_{\partial B_n(0,r)} \frac{1}{|x|^\alpha}\,dS(x)\,dr \\
&= \int_a^\infty \frac{1}{r^\alpha} n\alpha(n)r^{n-1}\,dr \\
&= \int_a^\infty n\alpha(n)r^{n-\alpha-1}\,dr \\
&= \frac{n\alpha(n)}{n-\alpha}r^{n-\alpha}\Big|_{r=a}^{r\to+\infty}.
\end{aligned}$$

We notice that this integral will only converge if $n - \alpha < 0$.

Now if $\alpha = n$, the integral becomes

$$\int_a^\infty n\alpha(n)r^{-1}\,dr = n\alpha(n)\ln r\big|_{r=a}^{r\to+\infty} \to +\infty.$$

Therefore, we conclude that this integral is finite if and only if

$$\boxed{\alpha > n.}$$

3. Find all radial solutions of

$$-\Delta u + u = 0 \qquad x \in \mathbb{R}^3 \setminus \{0\}.$$

Answer: Suppose there is a radial solution $u(x) = v(|x|)$. From class, we calculated that if $u(x) = v(|x|)$, then letting $r = |x|$,

$$\Delta u = \frac{n-1}{r}v'(r) + v''(r).$$

Therefore, in $n = 3$ dimensions, we have

$$\begin{aligned} -\Delta v + v = -v'' - \frac{2}{r}v' + v = 0 \\ \implies -rv'' - 2v' + rv = 0 \\ \implies -(rv)'' + rv = 0. \end{aligned}$$

Let $w = rv$. Therefore, we see that w satisfies

$$w'' - w = 0$$

which implies

$$w(r) = Ae^{r} + Be^{-r}.$$

Solving for v in terms of w, we have

$$v(r) = \frac{w}{r} = \frac{Ae^{r} + Be^{-r}}{r}.$$

Therefore, we see that all radial solutions of our equation must be given by

$$u(x) = v(|x|) = \frac{Ae^{|x|} + Be^{-|x|}}{|x|}.$$

4. Prove that

$$u(x) \equiv \frac{e^{-|x|}}{4\pi|x|}$$

satisfies

$$-\Delta u + u = \delta_0 \qquad x \in \mathbb{R}^3$$

in the sense of distributions.

Answer: Let $F_u : \mathcal{D}(\mathbb{R}^3) \to \mathbb{R}$ be defined such that

$$(F_u, \phi) = \int_{\mathbb{R}^3} u(x)\phi(x)\,dx \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^3).$$

To say that

$$-\Delta u + u = \delta_0$$

in the sense of distributions means that

$$(-\Delta F_u + F_u, \phi) = (\delta_0, \phi) = \phi(0).$$

Or, using the definition of derivative,

$$(F_u, -\Delta\phi) + (F_u, \phi) = \phi(0).$$

Therefore, we just need to show that

$$(F_u, -\Delta\phi) + (F_u, \phi) = \phi(0) \qquad \forall\, \phi \in \mathcal{D}(\mathbb{R}^3).$$

We write

$$\begin{aligned}(F_u, -\Delta\phi) &= -\int_{\mathbb{R}^3} u\Delta\phi\,dx \\ &= -\int_{B(0,\delta)} u\Delta\phi\,dx - \int_{K-B(0,\delta)} u\Delta\phi\,dx \\ &\equiv A + B\end{aligned}$$

where $K = \text{supp}(\phi)$. In addition we write

$$\begin{aligned}(F_u, \phi) &= \int_{B(0,\delta)} u\phi\,dx + \int_{K-B(0,\delta)} u\phi\,dx \\ &\equiv C + D.\end{aligned}$$

We will show that $A + B + C + D \to \phi(0)$ as $\delta \to 0^+$, and, thus, the desired result follows.

First, we look at terms A and C. We see that

$$\begin{aligned}|A + C| &= \left| -\int_{B(0,\delta)} u\Delta\phi\,dx + \int_{B(0,\delta)} u\phi\,dx \right| \\ &\leq \left(|\Delta\phi|_{L^\infty(B(0,\delta))} + |\phi|_{L^\infty(B(0,\delta))}\right) \int_{B(0,\delta)} |u|\,dx \\ &\leq C \int_{B(0,\delta)} \frac{e^{-|x|}}{|x|}\,dx \\ &\leq C \int_{B(0,\delta)} \frac{1}{|x|}\,dx.\end{aligned}$$

From our answer to 2(a), we know that this integral is equal to

$$\frac{3\alpha(3)}{2}\delta^2 = O(\delta^2).$$

Therefore,

$$|A + C| \to 0 \qquad \text{as } \delta \to 0^+.$$

We now need to look at terms B and D. Integrating B by parts, we have

$$-\int_{K-B(0,\delta)} u\Delta\phi\,dx = \int_{K-B(0,\delta)} \nabla u \cdot \nabla\phi\,dx - \int_{\partial B(0,\delta)} u\frac{\partial\phi}{\partial\tilde{\nu}}\,dS$$

where $\partial\phi/\partial\tilde{\nu} \equiv \nabla\phi \cdot \tilde{\nu}$ for $\tilde{\nu}$ the inner unit normal to $B(0,\delta)$. Above, we used the fact that $\phi \equiv 0$ on ∂K. We now need to look at the boundary term above. We have

$$\begin{aligned}\left| \int_{\partial B(0,\delta)} u\frac{\partial\phi}{\partial\nu}\,dS \right| &\leq C \left| \frac{\partial\phi}{\partial\nu} \right|_{L^\infty(\partial B(0,\delta))} \int_{\partial B(0,\delta)} \frac{e^{-|x|}}{|x|}\,dS \\ &\leq \frac{C}{|x|} \int_{\partial B(0,\delta)} dS \\ &= \frac{C}{\delta}\delta^{n-1} \\ &= C\delta \to 0 \text{ as } \delta \to 0^+.\end{aligned}$$

Therefore, we see that

$$-\int_{K-B(0,\delta)} u\Delta\phi\,dx = \int_{K-B(0,\delta)} \nabla u\cdot\nabla\phi\,dx + O(\delta).$$

Integrating by parts again, we have

$$\begin{aligned}-\int_{K-B(0,\delta)} u\Delta\phi\,dx &= -\int_{K-B(0,\delta)} \Delta u\phi\,dx + \int_{\partial B(0,\delta)} \frac{\partial u}{\partial\tilde{\nu}}\phi\,dS + O(\delta)\\ &\equiv B(1)+B(2).\end{aligned}$$

Now $B(1)+D=0$ by direct calculation, or using the result of problem 3. In particular, we know that $-\Delta(e^{-|x|}/|x|)+e^{-|x|}/|x|=0$ for $x\neq 0$. Finally, it remains to look at $B(2)$. By direct calculation we see that

$$\nabla u = -\frac{1}{4\pi}\left[\frac{e^{-|x|}x}{|x|^2}+\frac{e^{-|x|}x}{|x|^3}\right].$$

Further, we know that the inner unit normal on $\partial B(0,\delta)$ is given by

$$\tilde{\nu}=\frac{-x}{|x|}.$$

Therefore, we see that

$$\frac{\partial u}{\partial\tilde{\nu}} = \frac{1}{4\pi}\left[\frac{e^{-|x|}}{|x|}+\frac{e^{-|x|}}{|x|^2}\right].$$

Therefore, we have two terms to look at. First, we see that

$$\begin{aligned}\left|\frac{1}{4\pi}\int_{\partial B(0,\delta)}\frac{e^{-|x|}}{|x|}\phi\,dS\right| &= \left|\frac{e^{-\delta}}{4\pi\delta}\int_{\partial B(0,\delta)}\phi\,dS\right|\\ &\leq \frac{C}{\delta}\int_{\partial B(0,\delta)} dS\\ &= \frac{C}{\delta}\delta^2 = C\delta \to 0 \text{ as } \delta\to 0^+.\end{aligned}$$

Now for the second term we have

$$\begin{aligned}\frac{1}{4\pi}\int_{\partial B(0,\delta)}\frac{e^{-|x|}}{|x|^2}\phi\,dS &= \frac{e^{-\delta}}{4\pi\delta^2}\int_{\partial B(0,\delta)}\phi\,dS\\ &= e^{-\delta}⨍_{\partial B(0,\delta)}\phi\,dS \to e^{-0}\phi(0)=\phi(0) \text{ as } \delta\to 0^+,\end{aligned}$$

using the fact that ϕ is continuous. Combining all of the above estimates, we conclude that

$$A+B+C+D\to\phi(0) \qquad \text{as } \delta\to 0^+.$$