Math 220B - Summer 2003
Homework 5 Solutions

1. Let $\Omega = (0,k)\times(0,l)$. Use separation of variables to solve the following boundary-value problem for Laplace's equation on a square,

$$\begin{cases} \Delta u = 0 & (x,y) \in \Omega \\ u(0,y) = 0, u_x(k,y) = \phi(y) & 0 < y < l \\ u_y(x,0) = 0, u(x,l) = 0 & 0 < x < k. \end{cases}$$

Answer: Using separation of variables ($u = XY$) implies

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda.$$

We are led to the eigenvalue problem

$$\begin{cases} Y'' = -\lambda Y & 0 < y < l \\ Y'(0) = 0 = Y(l). \end{cases}$$

First, we look for positive eigenvalues $\lambda = \beta^2 > 0$. In particular, this implies

$$Y(y) = A\cos(\beta y) + B\sin(\beta y).$$

The boundary condition

$$Y'(0) = 0 \implies B = 0.$$

The boundary condition

$$Y(l) = 0 \implies A\cos(\beta l) = 0 \implies B = \frac{\left(n + \frac{1}{2}\right)\pi}{l} \qquad n = 0, 1, 2, \ldots$$

It is straightforward to check that these are all the eigenvalues.

Next, we need to solve

$$-X'' = -\lambda X.$$

Using the fact that $\lambda = \beta^2 > 0$, we see that

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

The boundary condition

$$X(0) = 0 \implies A = 0.$$

Therefore, $X(x) = B\sinh(\beta x)$. Therefore, we look for coefficients A_n such that a function of the form

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sinh(\beta_n x)\cos(\beta_n y)$$

will satisfy our boundary condition $u_x(k, y) = \phi(y)$. In particular, we need

$$\sum_{n=0}^{\infty} A_n \beta_n \cosh(\beta_n x) \cos(\beta_n y) = \phi(y),$$

which implies our coefficients must be given by

$$A_n \beta_n \cosh(\beta_n k) = \frac{\langle \cos(\beta y), \phi \rangle}{\langle \cos(\beta y), \cos(\beta y) \rangle}.$$

To summarize, our solution is given by

$$\boxed{u(x, y) = \sum_{n=0}^{\infty} A_n \sinh(\beta_n x) \cos(\beta_n y)}$$

where

$$\boxed{\beta_n = \frac{\left(n + \frac{1}{2}\right) \pi}{l}}$$

and

$$\boxed{A_n = \frac{1}{\beta_n \cosh(\beta_n k)} \frac{\langle \cos(\beta y), \phi \rangle}{\langle \cos(\beta y), \cos(\beta y) \rangle}.}$$

2. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Prove uniqueness of solutions of

$$\begin{cases} \Delta u = f & x \in \Omega \\ \frac{\partial u}{\partial \nu} + \alpha u = g & x \in \partial\Omega \end{cases}$$

for $\alpha > 0$.

Answer: Suppose there are two solutions u and v. Let $w = u - v$. Then w satisfies

$$\begin{cases} \Delta w = 0 & x \in \Omega \\ \frac{\partial w}{\partial \nu} + \alpha w = 0 & x \in \partial\Omega. \end{cases}$$

Therefore,

$$\begin{aligned} 0 &= \int_\Omega w \Delta w \, dx \\ &= -\int_\Omega |\nabla w|^2 \, dx + \int_{\partial\Omega} w \frac{\partial w}{\partial \nu} \, dS(x) \\ &= -\int_\Omega |\nabla w|^2 \, dx - \alpha \int_\Omega w^2 \, dS(x). \end{aligned}$$

Since $\alpha > 0$, the only way this equality can hold is if both terms on the right-hand side are identically zero. In particular, $|\nabla w| \equiv 0$ for $x \in \Omega$ and $w \equiv 0$ for $x \in \partial\Omega$. We conclude that $w \equiv 0$ throughout Ω.

3. Let $\Omega \equiv \{(x,y) : a^2 < x^2 + y^2 < b^2\}$ be an annular region in $\mathbb{R}^2$. Consider

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ \frac{du}{d\nu} + \alpha u = g(\theta) & x^2 + y^2 = a^2 \\ \frac{du}{d\nu} + \beta u = h(\theta) & x^2 + y^2 = b^2 \end{cases}$$

where ν is the outer unit normal to Ω.

(a) Solve this boundary-value problem in the case when $\alpha = \beta = 1$, $a = 1$, $b = 2$, $f(\theta) = 0$ and $g(\theta)$ is an arbitrary function.

Answer: Rewriting the equation in polar coordinates and using separation of variables, we see that

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

and

$$R_n(r) = \begin{cases} C_0 + D_0 \ln r & n = 0 \\ C_n r^n + D_n r^{-n} & n = 1, 2, \ldots \end{cases}$$

Now the boundary condition at $r = 2$ is $R'(2) + R(2) = 0$. Therefore, we have

$$R_0'(2) + R_0(2) = \frac{D_0}{2} + C_0 + D_0 \ln 2 = 0 \implies \boxed{C_0 = -D_0\left(\frac{1}{2} + \ln 2\right)}$$

and

$$R_n'(2) + R_n(2) = (nC_n 2^{n-1} - nD_n 2^{-n-1}) + (C_n 2^n + D_n 2^{-n}) = 0$$
$$\implies \boxed{C_n = \frac{D_n(n2^{-n-1} - 2^{-n})}{n2^{n-1} + 2^n}}.$$

Let

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n})(A_n \cos(n\theta) + B_n \sin(n\theta)).$$

Using the above relations on C_n and D_n, we have

$$\boxed{\begin{aligned} u(r,\theta) &= -D_0\left(\frac{1}{2} + \ln 2\right) + D_0 \ln r \\ &\quad + \sum_{n=1}^{\infty} \left(\left(\frac{n2^{-n-1} - 2^{-n}}{n2^{n-1} + 2^n} \right) r^n + r^{-n} \right) (A_n \cos(n\theta) + B_n \sin(n\theta)). \end{aligned}}$$

Now, we want $\partial u/\partial \nu + u = g(\theta)$ at $r = 1$. This implies

$$-u_r(1,\theta) + u(1,\theta) = -D_0\left(\frac{3}{2} + \ln 2\right) + \sum_{n=1}^{\infty} \gamma(n)(A_n \cos(n\theta) + B_n \sin(n\theta)) = g(\theta)$$

where

$$\gamma(n) = (1-n)\left(\frac{n2^{-n-1} - 2^{-n}}{n2^{n-1} + 2^n}\right) + (n+1)$$

Using the orthogonality of eigenfunctions, we conclude that

$$\boxed{\begin{aligned} -D_0\left(\frac{3}{2} + \ln 2\right) &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta \\ \gamma(n)A_n &= \frac{1}{\pi}\int_0^{2\pi} \cos(n\theta)g(\theta)\,d\theta \\ \gamma(n)B_n &= \frac{1}{\pi}\int_0^{2\pi} \sin(n\theta)g(\theta)\,d\theta. \end{aligned}}$$

Therefore, we conclude that our solution u is given as defined above with coefficients D_n, C_n, A_n, B_n as defined above.

(b) From the result from the previous problem, we know the solution to part (a) is unique. Prove that uniqueness may fail if either α or β are negative, by finding two solutions of

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x, y) \in \Omega \\ \frac{du}{d\nu} + 2u = 0 & x^2 + y^2 = 1 \\ \frac{du}{d\nu} - u = 0 & x^2 + y^2 = 4. \end{cases}$$

Answer: Clearly, $u = 0$ is a solution. Using the same technique as in the previous problem, our equation for R_n is given by

$$R_n = C_n r^n + D_n r^{-n}.$$

The boundary condition

$$\begin{aligned} \frac{\partial u}{\partial \nu} - u = 0 \text{ at } r = 2 &\implies R_n'(2) - R_n(2) = 0 \\ &\implies C_n(n2^{n-1} - 2^n) + D_n(-n2^{-n-1} - 2^{-n}) = 0. \end{aligned}$$

The boundary condition

$$\begin{aligned} \frac{\partial u}{\partial \nu} + 2u = 0 \text{ at } r = 1 &\implies -R_n'(1) + 2R_n(1) = 0 \\ &\implies C_n(n2^{n-1} - 2^n) + D_n(-n2^{-n-1} - 2^{-n}) = 0. \end{aligned}$$

These conditions imply that if $n = 2$, then C_n is arbitrary.
Therefore, we conclude that

$$\boxed{u(r, \theta) = r^2(A\cos(2\theta) + B\sin(2\theta))}$$

is a solution for A, B arbitrary.

4. (a) Find the one-dimensional Green's function for $\Omega = (0, l)$, That is, find the function $G(x, y)$ such that for each $x \in \Omega$,

$$\begin{cases} -\Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega. \end{cases}$$

You may use the fact that the fundamental solution of Laplace's equation in one dimension is $\Phi(x) = -\frac{1}{2}|x|$.

Answer: Let

$$G(x,y) = \Phi(y-x) - h^x(y)$$

where $\Phi(y)$ is the fundamental solution of Laplace's equation in $\mathbb{R}$; that is,

$$\Phi(y-x) = -\frac{1}{2}|y-x|.$$

and $h^x(y)$ is a solution of

$$\begin{array}{ll} \Delta_y h^x(y) = 0 & y \in (0,l) \\ h^x(y) = \Phi(y-x) = -\frac{1}{2}|y-x| & y = 0, l. \end{array}$$

We first solve for $h^x(y)$. Fix $x \in (0, l)$.

$$\Delta_y h^x(y) = 0 \implies h^x(y) = c_1 y + c_2$$

for c_1, c_2 arbitrary. The boundary conditions

$$\begin{aligned} h^x(0) &= -\frac{1}{2}|-x| = -\frac{1}{2}x \\ h^x(l) &= -\frac{1}{2}|l-x| = -\frac{1}{2}l + \frac{1}{2}x \end{aligned}$$

imply

$$\begin{aligned} c_1 &= -\frac{1}{2} + \frac{1}{l}x \\ c_2 &= -\frac{1}{2}x. \end{aligned}$$

Therefore,

$$h^x(y) = \left[-\frac{1}{2} + \frac{1}{l}x\right] y - \frac{1}{2}x.$$

Therefore,

$$\boxed{G(x,y) = -\frac{1}{2}|y-x| + \frac{1}{2}y - \frac{1}{l}xy + \frac{1}{2}x.}$$

We can rewrite this as

$$\boxed{G(x,y) = \begin{cases} x - \frac{1}{l}xy & y > x \\ y - \frac{1}{l}xy & y < x. \end{cases}}$$

(b) Use the Green's function above to solve the ODE

$$\begin{cases} u''(x) = 1 & x \in (0,1) \\ u(0) = 3 \\ u(1) = 2. \end{cases}$$

Answer: For $l = 1$, we see that

$$G(x,y) = \begin{cases} x - xy & y > x \\ y - xy & y < x. \end{cases}$$

We recall that if u is a solution of

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = g & x \in \partial\Omega, \end{cases}$$

then

$$u(x) = -\int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y) g(y)\, dS(y) + \int_{\Omega} G(x,y) f(y)\, dy.$$

Here

$$\frac{\partial G}{\partial \nu}(x,0) = -\frac{\partial G}{\partial y}(x,0) = x - 1$$
$$\frac{\partial G}{\partial \nu}(x,1) = \frac{\partial G}{\partial y}(x,1) = -x.$$

Therefore,

$$\begin{aligned} -\int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y) &= -\left[3\left[-\frac{\partial G}{\partial y}(x,0)\right] + 2\left[\frac{\partial G}{\partial y}(x,1)\right]\right] \\ &= -\left[3[x-1] + 2[-x]\right] \\ &= -x + 3. \end{aligned}$$

Now, using the fact that our inhomogeneous term $f(x) = -1$, we have

$$\begin{aligned} \int_{\Omega} f(y) G(x,y)\, dy &= \int_x^1 f(y) G(x,y)\, dy + \int_0^x f(y) G(x,y)\, dy \\ &= (-1)\int_x^1 [x - xy]\, dy + (-1)\int_0^x [y - xy]\, dy \\ &= \frac{x^2}{2} - \frac{x}{2}. \end{aligned}$$

Therefore, our solution u is given by

$$u(x) = -x + 3 + \frac{x^2}{2} - \frac{x}{2},$$

or

$$\boxed{u(x) = \frac{x^2}{2} - \frac{3x}{2} + 3.}$$

5. Find the Green's function for Laplace's equation on the half-ball $\Omega \equiv \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1, z > 0\}$.

   Answer: We can find the Green's function by taking the Green's function for the ball and reflecting it across the $xy-$plane. Let $x = (x_1, x_2, x_3)$. Define $x^* = x/|x|^2$, $\widetilde{x} = (x_1, x_2, -x_3)$, and $\widetilde{x^*} = \widetilde{x}/|x|^2$. Then, our Green's function can be written as

$$\boxed{G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)) - \Phi(y-\widetilde{x}) + \Phi(|x|(y-\widetilde{x^*})).}$$

6. Find the Green's function for Laplace's equation in the wedge $\Omega = \{(x_1,x_2) \in \mathbb{R}^2, x_2 > 0, x_1 > x_2\}$.

   Answer: We will use the method of reflection. Fix $x = (x_1, x_2) \in \Omega$. Let $z_1 = (x_1, -x_2)$ be the reflection of x about the the x_1 axis. Let $z_2 = (x_2, x_1)$ be the reflection of x about the line $x_1 = x_2$.

   Now, we need to continue in this way until we cancel out the boundary terms. In particular, let $z_3 = (x_2, -x_1)$, $z_4 = (-x_2, x_1)$, $z_5 = (-x_2, -x_1)$, $z_6 = (-x_1, -x_2)$, $z_7 = (-x_1, x_2)$.

   Then for $y \in \partial\Omega$ such that $y = (y_1, 0)$, we see that $|y-x| = |y-z_1|$, $|y-z_2| = |y-z_3|$, $|y-z_4| = |y-z_5|$, $|y-z_6| = |y-z_7|$.

   In addition, for $y = (y_1, y_2)$ in the other part of $\partial\Omega$, we see that $|y-x| = |y-z_2|$, $|y-z_1| = |y-z_4|$, $|y-z_3| = |y-z_7|$, $|y-z_5| = |y-z_6|$. Therefore, we get

$$\boxed{\begin{aligned} G(x,y) = \Phi(y-x) - \Phi(y-z_1) - \Phi(y-z_2) + \Phi(y-z_3) \\ + \Phi(y-z_4) - \Phi(y-z_5) + \Phi(y-z_6) - \Phi(y-z_7). \end{aligned}}$$