1. Consider the Neumann problem,
\[
\begin{aligned}
-\Delta u &= f & x &\in \Omega \\
\frac{\partial u}{\partial \nu} &= g & x &\in \partial \Omega
\end{aligned}
\]
Assume the compatibility condition holds. That is,
\[
-\int_\Omega f(x) \, dx = \int_{\partial \Omega} g(x) \, dS(x).
\]
Just as the Green’s function allowed us to find a representation formula for solutions to Poisson’s equation on a bounded domain \(\Omega\), here we construct a Neumann function to derive a representation formula for the Neumann problem. Let \(N(x, y)\) be defined as follows. Let
\[
N(x, y) = \Phi(y - x) - \tilde{h}^x(y) \quad \forall y \in \overline{\Omega}
\]
where \(\tilde{h}^x(y)\) is a solution of
\[
\begin{aligned}
\Delta_y \tilde{h}^x(y) &= 0 & \forall y &\in \Omega \\
\frac{\partial \tilde{h}^x}{\partial \nu}(y) &= \frac{\partial \Phi}{\partial \nu}(y - x) - C & \forall y &\in \partial \Omega
\end{aligned}
\]
for some appropriately chosen constant \(C\). (In part (b), you will determine the necessary constant for a given region \(\Omega\). For now, you may assume \(C\) is arbitrary.)

(a) Use \(N(x, y)\) to write a solution formula for
\[
\begin{aligned}
-\Delta u &= f & x &\in \Omega \\
\frac{\partial u}{\partial \nu} &= g & x &\in \partial \Omega
\end{aligned}
\]
in terms of \(f, g,\) and \(N\). (Note: As we know, Poisson’s equation with Neumann boundary conditions is only unique up to constants. Therefore, adding any constant to your solution formula will also give you a solution.)

(b) In the definition of \(\tilde{h}^x\), what must the constant \(C\) be? Explain.

2. (a) Find the Neumann function for \(\mathbb{R}^n_+\).
(b) Use the Neumann function for \(\mathbb{R}^n_+\) to find the solution formula for
\[
\begin{aligned}
\Delta u &= 0 & x &\in \mathbb{R}^n_+ \\
\frac{\partial u}{\partial \nu} &= g & x &\in \partial \mathbb{R}^n_+.
\end{aligned}
\]
3. Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$ with $C^2$ boundary. Let $h$ be a continuous function on $\partial \Omega$. Let $\Phi$ be the fundamental solution of Laplace’s equation on $\mathbb{R}^n$. Define the single-layer potential with moment $h$ as

$$
\overline{u}(x) = -\int_{\partial \Omega} h(y)\Phi(y - x)\,dS(y).
$$

(a) Show that $\overline{u}$ is defined and continuous for all $x \in \mathbb{R}^n$.

(b) Show that $\Delta \overline{u}(x) = 0$ for $x \notin \partial \Omega$.

4. Let $\Omega$ be an open, bounded set in $\mathbb{R}^n$ with smooth boundary. Let $\overline{\Omega} \equiv \mathbb{R}^n \setminus \Omega$. Consider the exterior Neumann problem,

$$
\begin{cases}
\Delta u = 0 & x \in \overline{\Omega}
\\frac{\partial u}{\partial \nu} = g & x \in \partial \overline{\Omega}.
\end{cases} \tag{*}
$$

Assume $g$ satisfies the condition,

$$
\int_{\partial \Omega} g(x)\,dS(x) = 0. \tag{**}
$$

(Note: Recall: This is not a necessary condition for solvability of the exterior Neumann problem.) Suppose a solution $u$ of $(*)$ is given by the single-layer potential,

$$
u(x) \equiv -\int_{\partial \Omega} h(y)\Phi(x - y)\,dS(y)
$$

where $h$ satisfies the integral equation

$$
g(x) = \frac{1}{2} h(x) - \int_{\partial \Omega} h(y)\frac{\partial \Phi(x - y)}{\partial \nu_x} \,dS(y).
$$

(a) Show that if $g$ satisfies the condition (**), then

$$
\int_{\partial \Omega} h(y)\,dS(y) = 0.
$$

(b) Show that the solution $u$ will have decay rate $O(|x|^{1-n})$. In particular, show $|u(x)| \leq C|x|^{1-n}$. Hint: By (a), write $u(x) = -\int_{\partial \Omega} h(y)[\Phi(x - y) - \Phi(x)]\,dS(y)$.

5. Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Let $\overline{\Omega} \equiv \mathbb{R}^n \setminus \overline{\Omega}$. Prove there exists at most one solution $u$ which decays to 0 as $|x| \to +\infty$ of the following

$$
\begin{cases}
\Delta u = f & x \in \overline{\Omega}
\u = g & x \in \partial \Omega.
\end{cases}
$$