Math 220B - Summer 2003
Homework 7
Due Thursday, August 14, 2003

1. Consider the eigenvalue problem,

$$\begin{cases} -\Delta w = \lambda w & x \in \Omega \\ \frac{\partial w}{\partial n} + a(x)w = 0 & x \in \partial\Omega. \end{cases} \tag{1}$$

   Let $\{v_i\}$ be the eigenfunctions for this problem. Let

$$Y_n \equiv \{w \in C^2 : w \not\equiv 0, \langle w, v_i \rangle = 0 \text{ for } i = 1, \ldots, n-1\}.$$

   Let

$$J(w) \equiv \left\{ \frac{\int_\Omega |\nabla w|^2 \, dx + \int_{\partial\Omega} a(x) w^2 \, dS(x)}{\int_\Omega w^2 \, dx} \right\}.$$

   Suppose there exists a function $u_n \in Y_n$ such that

$$J(u_n) = \min_{w \in Y_n} J(w).$$

   Let $m_n \equiv J(u_n)$. Show that m_n is the n^{th} eigenvalue of (1) with corresponding eigenfunction u_n.

2. Let $\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{4} < 1\right\}$. Consider the eigenvalue problem with Dirichlet boundary conditions,

$$\begin{cases} -\Delta u = \lambda u & (x,y) \in \Omega \\ u = 0 & (x,y) \in \partial\Omega. \end{cases} \tag{2}$$

   Compute the Rayleigh quotient of the trial function $w(x,y) = 4 - 4x^2 - y^2$ to approximate the first eigenvalue of (2). (*Hint: Make the substitution* $x = r\cos(\theta)$, $y = 2r\sin(\theta)$.)

3. Consider the eigenvalue problem (2). Let $w_1(x,y) = 4 - 4x^2 - y^2$. Let $w_2(x,y) = (4 - 4x^2 - y^2)^2$. Use the Rayleigh-Ritz approximation method to get an estimate on the first two eigenvalues of (2).

4. Consider the eigenvalue problem (2).

   (a) Find a lower bound on the first eigenvalue of (2) given by a rectangle containing Ω.

   (b) Find the best upper bound on the first eigenvalue of (2) given by rectangles inscribed within Ω with sides parallel to the x and y axes.

5. Let Ω be the ellipse given in problem 2. This time consider the eigenvalue problem with Neumann boundary conditions,

$$\begin{cases} -\Delta u = \lambda u & (x,y) \in \Omega \\ \frac{\partial u}{\partial \nu} = 0 & (x,y) \in \partial\Omega. \end{cases} \tag{3}$$

(a) Compute the Rayleigh quotient of $w(x, y) = y$.

(b) Prove that the Rayleigh quotient of w is a strict upper bound for the second eigenvalue of (3).

6. (a) Show that there does not exist a smooth function $f(x)$ with $f(0) = f(3) = 0$ and $\int_0^3 f'(x)^2\, dx = 1$, $\int_0^3 f(x)^2\, dx = 2$.

   (b) Find two linearly independent functions $f_1(x)$ and $f_2(x)$ satisfying $f_i(0) = f_i(3) = 0$ and $\int_0^3 f_i'(x)^2\, dx = 2$ and $\int_0^3 f_i(x)^2\, dx = 1$, $i = 1, 2$.