Mathematics Department Stanford University
Math. 285 Homework 1
Due at lecture Wednesday October 1

1. (i) If A_1, A_2 are non-empty compact subsets of $\mathbb{R}$, prove $\exists\, a_j \in A_j$, $j = 1, 2$, such that $|a_2 - a_1| \geq \frac{1}{2}(\mathcal{L}^1(A_1) + \mathcal{L}^1(A_2))$. ($\mathcal{L}^1$ denotes Lebesgue measure on $\mathbb{R}$.)

(ii) Using the result of (i) above prove that if $A \subset \mathbb{R}^n$ is compact and $j \in \{1, \ldots, n\}$ then $\operatorname{diam}(\mathcal{S}_j(A)) \leq \operatorname{diam}(A)$, where $\mathcal{S}_j(A)$ denotes the Steiner symmetrization of A as in the proof of Theorem 2.10.

2. Let $\mathcal{H}^m$ be m-dimensional Hausdorff (outer) measure on a metric space X, d. Prove that $\mathcal{H}^m$ is Borel regular.

Note: Evidently, as mentioned in lecture, $\mathcal{H}^m$ has the property that $\mathcal{H}^m(A \cup B) = \mathcal{H}^m(A) + \mathcal{H}^m(B)$ whenever $d(A, B) > 0$, so all Borel sets are $\mathcal{H}^m$-measurable by the Caratheodory theorem 1.13 of the text; thus for this question you merely need to check (directly from the definition of $\mathcal{H}^m$) that for every set $A \subset X$ there is a Borel set $B \supset A$ with $\mathcal{H}^m(B) = \mathcal{H}^m(A)$.

3. Let X, d be a metric space and let μ be a Borel-regular outer measure on X which is finite on each ball $B_\rho(x) \subset X$. In lecture we proved that the lower density $\Theta^n_*(\mu, x)(= \liminf_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n})$ is Borel measurable on X.

With a similar argument, prove that $\Theta^{*\,n}(\mu, x) = \limsup_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n}$ is also Borel measurable.

Hint: Start by proving that $\Theta^{*\,n}(\mu, x) = \limsup_{\rho \downarrow 0} \frac{\mu(\breve{B}_\rho(x))}{\omega_n \rho^n}$, where $\breve{B}_\rho(x)$ denotes the *open* ball of radius ρ and center x.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$. f is said to be approximately continuous at x with respect to Lebesgue measure if

$$\lim_{\rho \downarrow 0} \rho^{-n} \mathcal{L}^n(\{y \in B_\rho(x) : |f(y) - f(x)| \geq \varepsilon\}) = 0 \;\forall \varepsilon > 0.$$

Prove that if f is $\mathcal{L}^n$-measurable on $\mathbb{R}^n$ then f is approximately continuous $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

Hint: Lusin.

5. $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be approximately differentiable at $x \in \mathbb{R}^m$ with respect to Lebesgue measure if there is a linear map $v_x : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{\rho \downarrow 0} \rho^{-m} \mathcal{L}^m(\{y \in B_\rho(x) \setminus \{x\} : |y - x|^{-1} |f(y) - f(x) - v_x(y - x)| \geq \varepsilon\}) = 0 \;\forall \varepsilon > 0.$$

If f is locally Lipschitz near x (i.e. $\exists\, K, R > 0$ such that $|f(z) - f(y)| \leq K|z - y|$ for all $y, z \in B_R(x)$), prove that approximate differentiability at x is equivalent to differentiability at x.

(Recall differentiability at x means that there is a linear map $v_x : \mathbb{R}^m \to \mathbb{R}^n$ such that $\lim_{y \to x} |x - y|^{-1} |f(y) - f(x) - v_x(y - x)| = 0$; i.e. for each $\varepsilon > 0$ there is $\delta > 0$ such that $|x - y|^{-1} |f(y) - f(x) - v_x(y - x)| < \varepsilon$ for all $y \in B_\delta(x) \setminus \{x\}$.)