Mathematics Department Stanford University
Math. 285 Assignment 2
DUE AT LECTURE WEDNESDAY OCT 8

1. (Tietze extension theorem) Assume X is an arbitrary metric space, $A \subset X$ is closed, $\lambda > 0$, and $f : A \to \mathbb{R}$ is bounded continuous with $\sup_A |f| \le \lambda$.

(i) Let $A_+ = \{x : f(x) \ge \lambda/3\}$, $A_- = \{x : f(x) \le -\lambda/3\}$ and check that if $A_\pm$ are non-empty then $h_1(x) = \frac{\lambda}{3}(d(x, A_-) - d(x, A_+))/(d(x, A_+) + d(x, A_-))$ defines a continuous function $h_1 : X \to [-\lambda/3, \lambda/3]$ on X such that $h_1 \equiv \lambda/3$ on A_+ and $h_1 \equiv -\lambda/3$ on A_-; note also that if $A_- = \emptyset$ then such a function h_1 is obtained by taking $h_1 \equiv \lambda/3$.

(ii) Prove by (i) and induction on k that for each $k = 1, 2, \ldots$ there exist continuous $h_1, h_2, \ldots, h_k$ on X such that $\sup_A |f - \sum_{j=1}^k h_j| \le (2/3)^k \lambda$ and $|h_j| \le 2^{j-1}\lambda/3^j$ on X, $j = 1, 2, \ldots, k$.

(iii) By letting $k \to \infty$ in (ii), prove there is a continuous $H : X \to [-\lambda, \lambda]$ such that $H|A = f$.

(iv) Prove there is a continuous $H : X \to \mathbb{R}$ with $H|A = f$ even if no boundedness hypothesis is assumed for f.

Hint for (iv): Start by applying (iii) with $\arctan f$ in place of f. Caution: In this case (iii) would be applied with $\lambda = \pi/2$ and the extension H may possibly have a non-empty set $C = \{x : |H(x)| = \pi/2\}$, so you cannot get the required extension for f simply by using $\tan H$. (Note however that C is a closed set disjoint from A.)

2. Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous on $\mathbb{R}^m$ and satisfies $\limsup_{y\to x} |y - x|^{-1}|f(y) - f(x)| < \infty$ at almost all points $x \in \mathbb{R}^m$ (that is, $\lim_{\rho\downarrow 0} \sup_{0<|y-x|<\rho} |y - x|^{-1}|f(y) - f(x)|) < \infty$ a.e.)

(i) If $C_j = \{x : |f(y) - f(x)| \le j|y - x| \text{ whenever } |y - x| < 1/j\}$, prove that C_j is closed and that $\mathcal{L}^n(\mathbb{R}^n \setminus (\cup_j C_j)) = 0$.

(ii) Let $\cup_i C_{j,i}$ be a decomposition of C_j into closed (not necessarily disjoint) subsets of diameter $< 1/j$. Prove that $f|C_{j,i}$ is Lipschitz.

(iii) Prove that f is approximately differentiable (see Q.5 of hw1) at $\mathcal{L}^n$-a.e. point $x \in \mathbb{R}^m$

3. Let X, Y be metric spaces with X σ-compact and let $f : A \to Y$ be Lipschitz with A $\mathcal{H}^m$-σ-finite (i.e. $A = \cup_{j=1}^\infty A_j$ $\mathcal{H}^m(A_j) < \infty$ for each j) and $\mathcal{H}^m$-measurable. For each $B \subset A$ let $\mathcal{N}(B, f, y)$ ($y \in Y$) be the multiplicity function $\mathcal{H}^0(B \cap f^{-1}(y))$.

(i) Prove that $\mathcal{H}^m(\{y : \mathcal{N}(A_j, f, y) = \infty\}) = 0$ for each j.

(ii) Give an example to show that $\mathcal{H}^m(\{y : \mathcal{N}(A, f, y) = \infty\}) = \infty$ is possible with the stated hypotheses.

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and $\int_{\mathbb{R}^n} |Df| \le 1$. Prove that for each $K > 0$ the slices $\{x \in \mathbb{R}^n : f(x) = t\}$ have $(n-1)$-dimensional Hausdorff measure $\le K$ with the exception of a set of t of Lebesgue measure $\le 1/K$. Hint: Coarea formula.

5. Suppose X is a σ-compact metric space, $m \in \{1, 2, \ldots\}$, $k > 0$, $A \subset X$ is $\mathcal{H}^{m+k}$-measurable with $\mathcal{H}^{m+k}(A) < \infty$, and $f : A \to \mathbb{R}^m$ is Lipschitz. Prove $\mathcal{H}^k(f^{-1}y))$ is an $\mathcal{H}^m$-measurable function of $y \in \mathbb{R}^m$.

Hint: The key step (also the most difficult step) is the case when A is compact. Keep in mind that, in this case, if $y \in \mathbb{R}^m$ and U is an open subset of X containing $f^{-1}y$ then there is $\delta > 0$ such that $z \in B_\delta(y) \Rightarrow f^{-1}z \subset U$; then in particular check that, for $t \ge 0$ and $i = 1, 2, \ldots$, the sets V_i consisting of all points $y \in \mathbb{R}^m$ such that there is an open cover $U_{i1}, U_{i2}, \ldots$ of $f^{-1}y$ with $\operatorname{diam} U_{ij} < 1/i$ and $\sum_j \omega_k(\operatorname{diam} U_{ij}/2)^k < t + 1/i$ are *open* in $\mathbb{R}^m$. On the other hand, using the definition of $\mathcal{H}^k_{1/i}$, check that $V_i = \{y : \mathcal{H}^k_{1/i}(f^{-1}y) < t + 1/i\}$.