Mathematics Department Stanford University
Math. 285 Homework 3

Due at lecture Wednesday Oct 15

Note: References to the text assume a draft dated Oct 9 or later

1. Suppose $a, b \in \mathbb{R}$, $a < b$, and $\gamma : [a, b] \to \mathbb{R}^n$ is absolutely continuous. The length L of γ is defined as usual by $L = \int_a^b |\gamma'(t)|\, dt$. If $\widetilde{\gamma} : [0, L] \to \mathbb{R}^n$ is defined by $\widetilde{\gamma}(\tau) = \gamma(t(\tau))$, where $t(\tau) = \sup\{t \in [a, b] : \int_a^t |\gamma'(x)|\, dx \le \tau\}$ for $\tau \in [0, L]$. Prove (i) $\widetilde{\gamma}$ is Lipschitz with $\operatorname{Lip} \widetilde{\gamma} \le 1$, (ii) $|\widetilde{\gamma}'(s)| = 1$ for a.e. $s \in (0, L)$, (iii) $\widetilde{\gamma}([0, L]) = \gamma([a, b])$.

2. If $C \subset \mathbb{R}^2$ is the purely 1-unrectifiable subset constructed in Example 3.5 of Ch. 3 of the text, prove that $C \times [0, 1]$ has positive $\mathcal{H}^2$-measure and is purely 2-unrectifiable.

3. (i) If $v_1, \ldots, v_{n+\ell}$ is a basis for $\mathbb{R}^{n+\ell}$ and if L is an n-dimensional subspace of $\mathbb{R}^{n+\ell}$, prove that there exist $1 \le j_1 < j_2 < \cdots < j_n \le n+\ell$ such that the orthogonal projection $p_{v_{j_1},\ldots,v_{j_n}}$ of $\mathbb{R}^{n+\ell}$ onto $\operatorname{span}\{v_{j_1}, \ldots, v_{j_n}\}$ has the property that $p_{v_{j_1},\ldots,v_{j_n}} | L$ is an isomorphism of L onto $\operatorname{span}\{v_{j_1}, \ldots, v_{j_n}\}$.

Hint: You can of course assume without loss of generality that $L = \mathbb{R}^n \times \{0\}$. Observe that

$\operatorname{rank} p_{v_{j_1},\ldots,v_{j_n}} | \mathbb{R}^n \times \{0\} = \operatorname{rank} p_{v_{j_1},\ldots,v_{j_n}} \circ p_{\mathbb{R}^n \times \{0\}} = \operatorname{rank} p_{\mathbb{R}^n \times \{0\}} \circ p_{v_{j_1},\ldots,v_{j_n}} = \operatorname{rank} p_{\mathbb{R}^n \times \{0\}} | \operatorname{span}\{v_{j_1}, \ldots, v_{j_n}\}$

(ii) Using (i), check the claim made in Remark 3.4 of Ch. 3 of the text.

4. Justify the claim made in Remark 1.5(3) of Ch. 3 of the text, that if M is $\mathcal{H}^n$-measurable with $\mathcal{H}^n(M \cap K) < \infty$ for each compact K and if $x \in \mathbb{R}^{n+\ell}$ is such that the approximate tangent space $T_x M$ exists, then

$$\lim_{\rho \downarrow 0} \rho^{-n} \mathcal{H}^n(M \cap X_{1/2}((T_x M)^\perp, x) \cap B_\rho(x)) = 0.$$

5. If M is an n-dimensional C^1 submanifold of $\mathbb{R}^{n+\ell}$, if $x \in M$, and if $T_x M$ is the tangent space of M at x, prove that $T_x M$ is also the approximate tangent space of M at x; i.e.

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda} M} f \, d\mathcal{H}^n = \int_{T_x M} f \, d\mathcal{H}^n$$

for every $f \in C_c^0(\mathbb{R}^{n+\ell})$.

Hint: Suppose without loss of generality that $T_x M = \mathbb{R}^n \times \{0\}$ and $x = 0$, and use a local graphical representation for M near 0 as discussed in Remark 4.4 of Ch. 2 of the text.