# Mathematics Department Stanford University
# Math. 285 Homework 4

DUE AT LECTURE WEDNESDAY OCT 22

1. Let $\gamma_j : [a_j, b_j] \to \mathbb{R}^n$ be absolutely continuous and such that $\gamma_j(a_j) = 0$ and length $\gamma_j = 1$ for each $j = 1, 2, \ldots$.

(i) Prove there is a Lipschitz map $\gamma : [0, 1] \to \mathbb{R}^n$ with $\operatorname{Lip} f \le 1$ such that a subsequence of $\gamma_j([a_j, b_j]) \to \gamma([0, 1])$ in the Hausdorff distance sense.

Note on terminology: Given sets $A, B$ in a metric space $X$, the Hausdorff distance between $A, B$ is defined as the inf of the set of $\lambda > 0$ such that $A \subset \{x \in X : d(x, B) < \lambda\}$ and $B \subset \{x \in X : d(x, A) < \lambda\}$.

(ii) Construct an example of a sequence $\gamma_j$ which shows that $\gamma$ in (i) may have length strictly less than 1.

2. For each $N = 2, 3, \ldots$ let $M_N$ be the 2-dimensional $C^\infty$ submanifold of $\mathbb{R}^3$ defined by

$$M_N = \cup_{j,k \in \{0, \pm 1, \pm 2, \ldots\}} \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} : |x - (j/N, k/N)| = 1/N^2\}.$$

(Thus $M_N$ is a countable pairwise disjoint union of cylinders with axes parallel to the third coordinate axis.)

Prove that $\mathcal{H}^2 \llcorner M_N \to 2\pi \mathcal{L}^3$ (i.e. $2\pi$ times Lebesgue measure on $\mathbb{R}^3$) as $N \to \infty$; i.e. prove

$$\int_{M_N} f \, d\mathcal{H}^2 \to 2\pi \int_{\mathbb{R}^3} f \, d\mathcal{L}^3 \text{ for each } f \in C^0_c(\mathbb{R}^3).$$

Hint: First show that $N^{-2} \sum_{j,k=0,\pm1,\pm2,\ldots} \int_{\mathbb{R}} f(j/N, k/N, y)\, dy \to \int_{R^3} f \, d\mathcal{L}^3$ for each $f \in C^0_c(\mathbb{R}^3)$.

3. With $M_N$ as in Q.2 above, prove that $\int_{M_N} \omega \to 0$ as $N \to \infty$ for each continuous 2-form $\omega$ on $\mathbb{R}^3$ with compact support in $\mathbb{R}^3$.

Note: Here we use the usual definition of $\int_M \omega$ for a 2-dimensional oriented $C^1$ submanifold of $\mathbb{R}^3$ and a continuous 2-form $\omega = \omega_1 dx^2 \wedge dx^3 + \omega_2 dx^1 \wedge dx^3 + \omega_3 dx^1 \wedge dx^2$; namely, we assume that we have selected a continuous unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ for $M$ and then $\int_M \omega = \int_M \omega^* \cdot \nu \, d\mathcal{H}^2$, where $\omega^* = (\omega_1, -\omega_2, \omega_3)$ the vector field dual to $\omega$.

4. Let $M$ be a smooth $n$-dimensional minimal surface in $\mathbb{R}^{n+\ell}$ with $\overline{M} \setminus M = \emptyset$, $0 \in M$ and $\lim_{\rho \to \infty} (\omega_n \rho^n)^{-1} \mathcal{H}^n(M \cap B_\rho(0)) = 1$. Prove that $M$ is an $n$-dimensional subspace of $\mathbb{R}^{n+\ell}$.

Hint: Monotonicity identity.

5. Suppose $M$ is a bounded, $\mathcal{H}^n$-measurable, countably $n$-rectifiable subset of $\mathbb{R}^{n+\ell}$ with $\mathcal{H}^n(M \cap K) < \infty$ for each compact $K \subset \mathbb{R}^{n+\ell}$ and with $M$ stationary in $\mathbb{R}^{n+\ell}$ (i.e. $\int_M \operatorname{div}_M X \, d\mathcal{H}^n = 0$ for each $C^1$ vector field $X$ on $\mathbb{R}^{n+\ell}$ with compact support in $\mathbb{R}^{n+\ell}$). Prove $\mathcal{H}^n(M) = 0$ (so in particular there are no smooth compact minimal surfaces without boundary in $\mathbb{R}^{n+\ell}$).

Hint: Use monotonicity.