

Mathematics Department Stanford University

Math. 285 Homework 5

DUE AT LECTURE WEDNESDAY OCT 29

1. Let $\gamma, t(\tau), \tilde{\gamma}$ be as in Q.1 of hw3.

(i) If $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, prove that the change of variable formula

$$\int_a^b h(\gamma(x)) \cdot \gamma'(x) dx = \int_0^L h(\tilde{\gamma}(\tau)) \cdot \tilde{\gamma}'(\tau) d\tau$$

is valid at least when $\exists \varepsilon > 0$ with $|\gamma'(x)| \geq \varepsilon$ for a.e. $x \in [a, b]$.

Hint: Show in this case that $\tau \in [0, L] \mapsto t(\tau) \in [a, b]$ is a 1:1, increasing, Lipschitz map of $[0, L]$ onto $[a, b]$, and hence (by the 1-dimensional area formula) we have $\int_a^b f(x) dx = \int_0^L f(t(\tau)) t'(\tau) d\tau$ for each $L^1([a, b])$ function f .

(ii) Prove the formula in (i) is valid without the assumption $|\gamma'(x)| \geq \varepsilon$.

Hint: To start, apply (i) to $\gamma_\varepsilon(t) = (\gamma(t), \varepsilon t) : [a, b] \rightarrow \mathbb{R}^{n+1}$ and note that $\tilde{\gamma}_\varepsilon(\tau) \rightarrow \tilde{\gamma}(\tau)$ on $[0, L]$ —because if $t_\varepsilon(\tau), \tilde{\gamma}_\varepsilon(\tau), L_\varepsilon$ are the analogues of $t(\tau), \tilde{\gamma}(\tau), L$ when we use γ_ε in place of γ , then, as $\varepsilon \downarrow 0$, $L_\varepsilon \downarrow L$ and $t_\varepsilon(\tau)$ increases to some limit $s(\tau)$ with $\text{length}(\gamma[[a, s(\tau)]] = \tau$, hence $s(\tau) \leq t(\tau)$, $\gamma[[s(\tau), t(\tau)]] = \text{const.}$, and $\tilde{\gamma}_\varepsilon(\tau) = \gamma_\varepsilon(t_\varepsilon(\tau)) \rightarrow \gamma(s(\tau)) = \gamma(t(\tau)) = \tilde{\gamma}(\tau)$.

2. Using the result of Q.1 (ii) above, prove that if $\gamma_j : [a_j, b_j] \rightarrow \mathbb{R}^n$ is a sequence as in Q.1 of hw4, then there is a Lipschitz $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ and a subsequence $\gamma_{j'}$ such that

$$\int_{a_{j'}}^{b_{j'}} \langle \omega|_{\gamma_{j'}(x)}, \gamma_{j'}'(x) \rangle dx \rightarrow \int_0^1 \langle \omega|_{\gamma(x)}, \gamma'(x) \rangle dx$$

for each continuous 1-form $\omega = \sum_{i=1}^n \omega_i dx^i$ on \mathbb{R}^n , where $\langle \omega, v \rangle$ denotes the dual pairing between 1-forms and vectors in \mathbb{R}^n .

Note: The above is a version of the 1-dimensional case of the Federer-Fleming compactness theorem for integral currents.

3. Let U be open in $\mathbb{R}^{n+\ell}$, let $V = \underline{v}(M, \theta)$ be a rectifiable n -varifold which is stationary in U , let μ_V be the weight measure (i.e. $d\mu_V = \theta d\mathcal{H}^n \llcorner M$), and assume $\theta \geq 1$ μ_V -a.e.

(i) Prove $V = \underline{v}(S, \Theta)$, where $S = \text{spt } \mu_V$ (closed), and $\Theta(x) = \Theta^n(\mu_V, x)$, $x \in S$ (i.e. prove that $\mu_V((S \setminus M) \cup (M \setminus S)) = 0$ and $\Theta = \theta$ μ_V -a.e. on $M \cap S$).

(Recall that by definition $\text{spt } \mu_V = \{x \in U : \mu_V(B_\sigma(x) \cap U) > 0 \forall \sigma > 0\}$.)

(ii) If $B_\rho(y) \subset U$ and $(\omega_n \rho^n)^{-1} \mu_V(B_\rho(y)) < 1$, prove that $y \notin S$.

4. If U, V, S are as in Q.3 above, if $x \in U$, $q \in \{1, 2, \dots\}$, and if V has multiplicity q approximate tangent space at x (i.e. there is an n -dimensional subspace $L \subset \mathbb{R}^{n+\ell}$ such that $\int_{\eta_{x,\lambda} S} f \theta \circ \eta_{x,\lambda}^{-1} d\mathcal{H}^n \rightarrow q \int_L f d\mathcal{H}^n$ as $\lambda \downarrow 0 \forall f \in C_c^0(U)$), prove that $\eta_{x,\lambda} S \rightarrow L$ locally in the Hausdorff distance sense.

Note: $\eta_{x,\lambda} S \rightarrow L$ locally in the Hausdorff distance sense means that for each $R > 0$, $\varepsilon > 0$ there is $\delta > 0$ with $\eta_{x,\lambda} S \cap B_R(0) \subset \{x : \text{dist}(x, L) < \varepsilon\}$ and $L \cap B_R(0) \subset \{x : \text{dist}(x, \eta_{x,\lambda} S) < \varepsilon\}$ for all $0 < \lambda < \delta$.