

Mathematics Department Stanford University

Math. 285 Homework 6

DUE AT LECTURE WEDNESDAY NOV 5

1. Let V be a convex open subset of \mathbb{R}^Q , U is open in \mathbb{R}^P , $f, g : U \rightarrow V$ are proper C^∞ maps, and $h : [0, 1] \times U \rightarrow V$ is the affine homotopy $h(t, x) = tg(x) + (1 - t)f(x)$.

(i) Show that h is not proper if $g(x) = -f(x)$ for each $x \in U$.

(ii) If, in addition to the assumptions above, U is bounded and $f = \bar{f}|U, g = \bar{g}|U$ with \bar{f}, \bar{g} continuous on \bar{U} , prove that h is proper if $\bar{f} = \bar{g}$ on ∂U .

Hint: By definition of proper (i.e. the preimage of each compact set in V is a compact set in U), $f : U \rightarrow V$ is proper if and only if the following property holds: Whenever $\{x_k\} \subset U$ with either $|x_k| \rightarrow \infty$ or $\text{dist}(x_k, \partial U) \rightarrow 0$, then either $|f(x_k)| \rightarrow \infty$ or $\text{dist}(f(x_k), \partial V) \rightarrow 0$.

2. In lecture we proved that if $T \in \mathcal{D}_n(U)$ with $\mathbb{M}(T) < \infty, \mathbb{M}(\partial T) < \infty$, if $f, g : U \rightarrow V$ are C^1 maps such that $f|_{\text{spt } T}, g|_{\text{spt } T}$ are proper, and if $f|_{\text{spt } T} = g|_{\text{spt } T}$, then $f\#T = g\#T$.

Give an example to show that this may fail without the condition $\mathbb{M}(\partial T) < \infty$.

Hint: Let $T \in \mathcal{D}_1(\mathbb{R})$ be defined by $T(\omega) = a(0)$ for any 1-form $\omega = a dx^1 \in \mathcal{D}^1(\mathbb{R})$.

3. Check the claim 2.20 of Ch.6 of the text: That by applying 2.17, 2.18, one can check that $\mathbb{M}_{W_1 \times W_2}(S \times T) = \mathbb{M}_{W_1}(S)\mathbb{M}_{W_2}(T)$, assuming $S, T \in \mathcal{D}_n(U)$ have locally finite mass in U .

4. Suppose $R \in \mathcal{D}_n(\mathbb{R}^n)$ (i.e. we are in the setting $P = n$ and $U =$ the whole Euclidean space), and suppose R is an integer multiplicity current of finite mass.

(i) Prove that there are pairwise disjoint Lebesgue measurable subsets $U_j, j = \pm 1, \pm 2, \dots$, of \mathbb{R}^n such that $R = \sum_{j=1}^{\infty} j \llbracket U_j \rrbracket - \sum_{j=1}^{\infty} j \llbracket U_{-j} \rrbracket$ and $\mathbb{M}(R) = \sum_{j=1}^{\infty} j(\mathcal{L}^n(U_j) + \mathcal{L}^n(U_{-j}))$.

(ii) If $V_j = \cup_{k=j}^{\infty} U_k$ for $j = 1, 2, \dots$ and $W_j = \cup_{k=1+j}^{\infty} U_{-k}$ for $j = 0, 1, \dots$ (note that then $V_{j+1} \subset V_j$ and $W_j \subset W_{j-1}$ for each $j = 1, 2, \dots$), prove that $R = \sum_{j=1}^{\infty} \llbracket V_j \rrbracket - \sum_{j=0}^{\infty} \llbracket W_j \rrbracket$ and $\mathbb{M}(R) = \sum_{j=1}^{\infty} (\mathcal{L}^n(V_j) + \mathcal{L}^n(W_j))$.

Hint: If $a_j \geq 0$ for each $j = 1, 2, \dots$ and $b_i = \sum_{j=i}^{\infty} a_j$, then $\sum_{i=1}^{\infty} b_i = \sum_{j=1}^{\infty} j a_j$.

(iii) If $V_j, j = 1, 2, \dots, W_j, j = 0, 1, 2, \dots$ are as in (ii), and if $V_{-j} = \mathbb{R}^n \setminus W_j$ for $j = 0, 1, 2, \dots$ (so now $V_{j+1} \subset V_j$ for all $j = 0, \pm 1, \dots$) prove that $\partial R = \sum_{j=-\infty}^{\infty} \partial \llbracket V_j \rrbracket$ (and the sum makes sense—i.e. $\omega \in \mathcal{D}^{n-1}(\mathbb{R}^n) \Rightarrow \sum_{j=-\infty}^{\infty} \llbracket V_j \rrbracket(d\omega)$ is a convergent series).

5. If $T \in \mathcal{D}_{n-1}(\mathbb{R}^n)$ is an integer multiplicity current with $\text{spt } T$ compact, $\mathbb{M}(T) < \infty$ and $\partial T = 0$, prove that the cone $0 \times T$ (defined as in §2 of Ch.6 of the text) is also integer multiplicity of finite mass. Hence, using the result of Q.4, prove that there is a sequence $V_j, j = 0, \pm 1, \pm 2, \dots$, of Lebesgue measurable sets with $V_{j+1} \subset V_j$ for each j such that $T = \sum_{j=-\infty}^{\infty} \partial \llbracket V_j \rrbracket$ and $\mathbb{M}(T) \leq \sum_j \mathbb{M}(\partial \llbracket V_j \rrbracket)$.

Remark: Actually equality holds in the last inequality, i.e. $\mathbb{M}(T) = \sum_j \mathbb{M}(\partial \llbracket V_j \rrbracket)$, but we are not quite in a position to prove this because we skipped the discussion of sets of locally finite perimeter (§4 of Ch.3 of the text).

Suggestion (not part of the assignment problem): Take a look at a few examples of the case when $T = \gamma\#([0, 1])$, where γ is a C^1 immersion of $[0, 1]$ into \mathbb{R}^2 with $\gamma(0) = \gamma(1)$ (which ensures $\partial T = 0$); it is interesting to see how the sets V_j work out in such cases, when γ has a few self-intersections.