

Mathematics Department Stanford University

Math. 285 Homework 7

DUE AT LECTURE WEDNESDAY NOV 12

1. (Degree of a mapping.) Suppose U is bounded open in \mathbb{R}^n and $f : \bar{U} \rightarrow \mathbb{R}^n$ is continuous with $f|_U \in C^1(U, \mathbb{R}^n)$, let $\tilde{U} = U \setminus f^{-1}(f(\partial U)) \neq \emptyset$, and $\tilde{f} = f|_{\tilde{U}}$. Also, let W_1, W_2, \dots, W_N or W_1, W_2, \dots denote the connected components of $\mathbb{R}^n \setminus f(\partial U)$ (the latter alternative in the event there are infinitely many connected components).

(i) Prove that $\tilde{f} : \tilde{U} \rightarrow \mathbb{R}^n \setminus f(\partial U)$ is proper.

(ii) Prove $\tilde{f}_\#[\tilde{U}] = \sum_j d_j [W_j]$, where (a) $d_j \in \mathbb{Z}$ for each j , and (b) $W_j \not\subset f(U) \Rightarrow d_j = 0$. (Notice that (b) says $d_j \neq 0 \Rightarrow W_j \subset f(U)$.)

(iii) Prove furthermore that (a) $|d_j| \leq \mathcal{H}^0(f^{-1}y)$ for \mathcal{L}^n -a.e. $y \in W_j$, (b) $|d_j|$ is congruent to $\mathcal{H}^0(f^{-1}y) \pmod{2}$ for \mathcal{L}^n -a.e. $y \in W_j$, and (c) $d_j = \mathcal{H}^0(f^{-1}y)$ for any point $y \in W_j$ such that $J_f(x) > 0$ for each $x \in f^{-1}y$.

Hint: $\tilde{f}_\#[\tilde{U}](\omega) = \int_{\tilde{U}} \langle df_x^\# \omega|_{f(x)}, e_1 \wedge \dots \wedge e_n \rangle d\mathcal{L}^n = \int_{\tilde{U}} \langle \omega|_{f(x)}, df_{x\#}(e_1 \wedge \dots \wedge e_n) \rangle d\mathcal{L}^n$; compute using area formula.

Note: For $y \in W_j$, d_j is called the (topological) degree, denoted $d(f, U, y)$, of the map f at y .

2. (Invariance of degree under homotopy.)

(i) If U, f, W_j, d_j are as in Q.1, K compact $\supset f(\partial U)$, and $V = U \setminus f^{-1}K \neq \emptyset$, then $f_\#[V] = \sum_i c_i [E_i]$, where E_i are the connected components of $\mathbb{R}^n \setminus K$, and each $E_i \subset$ some W_j with $c_i = d_j$.

(ii) If U is bounded open in \mathbb{R}^n and $h : [0, 1] \times \bar{U} \rightarrow \mathbb{R}^n$ is continuous, if $f_t : U \rightarrow \mathbb{R}^n$ is C^1 for each $t \in [0, 1]$, with $f_t(x) = h(t, x)$ for $x \in U$, prove that $d(f_0, U, y) = d(f_1, U, y) \forall y \in \mathbb{R}^n \setminus h([0, 1] \times \partial U)$.

Hint: Show $d(f_t, U, y)$ is a continuous function of t for $y \in \mathbb{R}^n \setminus h([0, 1] \times \partial U)$.

3. Let $P(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$, where a_0, \dots, a_{n-1} are given complex constants. Prove, using Q.1,2 above, the ‘‘Fundamental Theorem of Algebra’’ that $P(z) = 0$ has a root (hence n roots by the Remainder Theorem and induction on n).

Hint: View P as a map of $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ and let $h(t, x, y) = z^n + t \sum_{j=0}^{n-1} a_j z^j$, $z = x + iy$, viewed as a map $[0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$; take $U = \{(x, y) : |(x, y)| < R\}$ (i.e. $\{z : |z| < R\}$), with R large.

4. (i) (Constancy theorem on a submanifold.) Suppose M is a connected oriented n -dimensional submanifold of \mathbb{R}^P and let $U \subset \mathbb{R}^P$ be open with $M \subset U$. If $T \in \mathcal{D}_n(U)$ with $\text{spt } T \subset M$ and $\partial T = 0$, prove that $T = c[M]$ for some real constant c .

Hint: Use local coordinates and the constancy theorem (proved in lecture) in \mathbb{R}^n .

(ii) (Degree of a map between n -dimensional manifolds.) If M, N are compact oriented connected C^1 submanifolds (without boundary) of \mathbb{R}^P and \mathbb{R}^Q respectively, and if $f : M \rightarrow N$ is C^1 , prove $f_\#[M] = d[N]$ for some $d \in \mathbb{Z}$.

Note: $d = d(f)$ is called the degree of the map $f : M \rightarrow N$.

(iii) If M, N are as in (ii) and if $h : [0, 1] \times M \rightarrow N$ is continuous with $f_t = M \rightarrow N$ of class C^1 for each $t \in [0, 1]$, where $f_t(x) = h(t, x)$ for $x \in M$, prove that $d(f_0) = d(f_1)$.

5. Give an example to show that the Federer-Fleming compactness theorem (Theorem 3.11 of Ch.6 of the text) fails without the hypothesis that $\sup_j \mathbb{M}_W(\partial T_j) < \infty$.

Hint: Modify the example in Q.3 of hw4; instead of vertical cylinders use the vertical strips $\{(x, y, z) \in \mathbb{R}^3 : |x - i/N| < 1/N^2, y = j/N, z \in \mathbb{R}\}$ with orienting 2-vector $e_1 \wedge e_3$, $i, j \in \{0, \pm 1, \pm 2, \dots\}$.

Note: Even better, deleting the z coordinate in the same example (so the vertical strips become line segments with orienting 1-vector e_1) gives a 1-dimensional sequence in \mathbb{R}^2 .