

**Mathematics Department Stanford University**

**Math. 285 Homework 8 (Final hw.)**

DUE AT LECTURE WEDNESDAY NOV 19

1. Let  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$  be defined by  $\gamma(r) = r(\cos((\log(2/r))^\alpha), \sin((\log(2/r))^\alpha))$  for  $r \in (0, 1]$  and  $\gamma(0) = (0, 0)$ , where  $\alpha \in (0, \frac{1}{2})$ . Prove

(i)  $|\gamma'(r)| \leq 2$  and hence  $\mathcal{H}^1(\gamma([0, 1])) \leq 2$ .

(ii) If  $\Gamma = \gamma([0, 1]) \cup (-\gamma)([0, 1])$ , prove that the approximate tangent space  $T_0\Gamma$  does not exist, but that  $\Gamma$  has the strong affine approximation property at 0, meaning that for each  $\sigma \in (0, 1]$  there is a 1-dimensional subspace  $T_\sigma$  of  $\mathbb{R}^2$  with  $\sigma^{-1} \text{dist}(T_\sigma \cap B_\sigma(0), \Gamma \cap B_\sigma(0)) \rightarrow 0$  as  $\sigma \downarrow 0$ .

Here  $\text{dist}$  means as usual Hausdorff distance  $d(A, B) = \inf$  of all numbers  $\lambda > 0$  such that  $A$  is contained in the  $\lambda$ -nhd. of  $B$  and  $B$  is contained in the  $\lambda$ -nhd. of  $A$ .

2. With  $\Gamma$  as above, calculate  $p_{(T_x\Gamma)^\perp}(x)$  for  $x \in \Gamma \setminus \{0\}$  and check  $\int_\Gamma r^{-3} |p_{(T_x\Gamma)^\perp}(x)|^2 d\mathcal{H}^1 < \infty$ .

Note: Q.1,2 suggest that finiteness of the term  $\int_\Gamma r^{-3} |p_{(T_x\Gamma)^\perp}(x)|^2 d\mathcal{H}^1$  (which is one of the key terms appearing in the monotonicity identity) does not in itself guarantee any especially strong regularity properties.

3. Let  $F(p) = \sqrt{\det(\delta_{ij} + p_i \cdot p_j)}$ , where  $p = (p_i^\alpha)_{i=1, \dots, n, \alpha=1, \dots, \ell} \in \mathbb{R}^{n\ell}$  and  $p_i = (p_i^1, \dots, p_i^\ell) \in \mathbb{R}^\ell$ .

(i) Prove that there is  $\varepsilon = \varepsilon(n, \ell) > 0$  such that  $F(p)$  is a convex function of  $p$  for  $|p| \leq \varepsilon$ .

(ii) Suppose  $u : \check{B}_\rho^n(0) \rightarrow \mathbb{R}^\ell$  is Lipschitz with  $\text{Lip } u \leq \varepsilon$  and let  $\mathcal{A}(u) = \mathcal{H}^n(\text{graph } u)$ . Prove that in fact  $\mathcal{A}(u) = \int_{\check{B}_\rho^n(0)} F(Du) d\mathcal{L}^n$ , and, if  $v : \check{B}_\rho^n(0) \rightarrow \mathbb{R}^\ell$  is also Lipschitz with  $\text{Lip } v \leq \varepsilon$ ,  $\mathcal{A}(u) - \mathcal{A}(v) \leq \int_{\check{B}_\rho^n(0)} \sum_{i, \alpha} A_i^\alpha(Du) D_i(u^\alpha - v^\alpha)$ , where  $A_i^\alpha(p) = \partial F(p) / \partial p_i^\alpha$ .

Hint: Let  $f(t) = \mathcal{A}(u + t(v - u))$ ,  $t \in [0, 1]$ , and use the 2nd order Taylor expansion  $f(1) = f(0) + f'(0) + \int_0^1 (1-t) f''(t) dt$  together with (i).

4.  $u$  as in Q.3(ii) is said to be a weak solution of the minimal surface system (MSS) if it is a weak solution of the Euler-Lagrange system for the functional  $\mathcal{A}(u)$ ; that is  $\frac{d}{ds} \big|_{s=0} \mathcal{A}(u + s\zeta) = 0$  for each Lipschitz  $\zeta$  with compact support in  $\check{B}_\rho^n(0)$ .

(i) Prove that this is exactly the requirement that  $\sum_{i, \alpha} \int_{\check{B}_\rho^n(0)} A_i^\alpha(Du) D_i \zeta^\alpha d\mathcal{L}^n = 0$  for for each Lipschitz  $\zeta$  with compact support in  $\check{B}_\rho^n(0)$ .

(ii) Prove using Q.3(ii) that if  $u$  is a Lipschitz weak solution of the MSS as in (i) with  $\text{Lip } u \leq \varepsilon$ , then  $\mathcal{A}(u) \leq \mathcal{A}(v)$  for every Lipschitz  $v : \check{B}_\rho^n(0) \rightarrow \mathbb{R}^\ell$  which is such that  $v - u$  has compact support in  $\check{B}_\rho^n(0)$  and  $\text{Lip } v \leq \varepsilon$ .

(iii) If  $u$  is as in (ii) except that now  $\text{Lip } u \leq \varepsilon/2$ , prove that  $G = \text{graph } u$  (viewed as a multiplicity 1 rectifiable varifold in  $\check{B}_\rho^n(0) \times \mathbb{R}^\ell$ ) is stationary; i.e., prove that if  $\varphi_t(x) = x + tX|_x$  for  $x \in \check{B}_\rho^n(0) \times \mathbb{R}^\ell$  with  $X = (X^1, \dots, X^{n+\ell}) \in C^1$  with compact support in  $\check{B}_\rho^n(0) \times \mathbb{R}^\ell$ , then  $\frac{d}{dt} \big|_{t=0} \mathcal{H}^n(\varphi_t(G)) = 0$ .

Hint: Show that, for small enough  $t$ ,  $\varphi_t(G)$  is again the graph of a Lipschitz function  $u_t$  with  $\text{Lip } u_t < \varepsilon$ , and then use (ii).

Note (not part of the assignment): Having proved (iii), we can immediately apply the Allard regularity theorem to deduce that  $u$  is  $C^{1, \alpha}(\check{B}_{(1-\theta)\rho}(0))$  for any  $\alpha, \theta \in (0, 1)$  provided  $\varepsilon = \varepsilon(\alpha, n, \ell, \theta)$  is small enough.