1. (10 points) Find each of the following limits, with justification. If there is an infinite limit, then explain whether it is \( \infty \) or \(-\infty\).

(a) \[ \lim_{x \to \frac{3\pi}{2}} \left( x - \frac{3\pi}{2} \right) \tan x \]

\( \left( \frac{\infty}{\infty} \right) \) form (because \( \tan x \to \infty \) as \( x \to \frac{3\pi}{2} \))

\[ \lim_{x \to \frac{3\pi}{2}} \frac{(x-\frac{3\pi}{2})\sin x}{\cos x} \quad \left( \frac{0}{0} \text{ form} \right) \]

(by L'Hôpital)

\[ = \lim_{x \to \frac{3\pi}{2}} \frac{\sin x + (x-\frac{3\pi}{2})\cos x}{-\sin x} \]

\[ = \frac{-1 + 0}{1} = -1 \]

(b) \[ \lim_{t \to -\infty} (\sqrt{t^2 - 5t} - t) \quad \left( \frac{\infty}{\infty} \right) \) form

\[ = \lim_{t \to -\infty} \frac{(t^2 - 5t - t)(\sqrt{t^2 - 5t} + t)}{t^2 - 5t + t} \]

\[ = \lim_{t \to -\infty} \frac{(t^2 - 5t) - t^2}{\sqrt{t^2 - 5t} + t} \]

\[ = \lim_{t \to -\infty} \frac{-5t}{\sqrt{t^2 - 5t} + t} \quad \left( \frac{\infty}{\infty} \text{ form} \right) \]

\[ = \lim_{t \to -\infty} \frac{(-5t) \cdot \frac{1}{t}}{(\sqrt{t^2 - 5t} + t) \frac{1}{t}} \]

\[ = \lim_{t \to -\infty} \frac{-5}{t^2 - 5t + 1} = \lim_{t \to -\infty} \frac{-5}{\sqrt{t^2} - \frac{5}{2} + 1} = \frac{-5}{\sqrt{1} + 1} = \left[ -\frac{5}{2} \right] \]

(Note: L'Hôpital is less advisable than dividing by \( t \), as it can create a mess.)
2. (8 points) Suppose \( c \) is a constant, and let \( g(x) \) be the function
\[
g(x) = \begin{cases} \frac{2}{x} + c^2 x & 0 < x < 1 \\ cx + 2c & x \geq 1. \end{cases}
\]

(a) Determine all values of \( c \) for which \( g(x) \) is continuous for all \( x > 0 \). Explain your reasoning.

Being rational or polynomial expressions, the pieces of \( g \) (on \((0,1)\) and \((1,\infty)\)) are individually continuous on their respective domains, so we know that the only issue is whether \( g \) is continuous at \( x = 1 \); i.e., whether \( \lim_{x \to 1^-} g(x) = g(1) = 3c \); in particular, the limit in question must exist. (And it will exist if its one-sided limits are equal.) So the key issue is that
\[
\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} cx + 2c = 3c \quad \text{must be equal to} \quad \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \frac{2}{x} + c^2 x = 2 + c^2,
\]
so that \( c^2 - 3c + 2 = 0 \). Thus \((c-2)(c-1) = 0\), so \([c = 1 \text{ or } c = 2]\); in each case we find that \( \lim_{x \to 1} g(x) \) exists and equals \( g(1) = 3c \), giving a continuous function \( g \).

(b) Determine all values of \( c \) for which \( g'(x) \) is continuous for all \( x > 0 \). Explain your reasoning.

Again, although \( g \) is piecewise a polynomial or rational expression, there are multiple ways for \( g' \) to fail to be continuous at a point \( x = a \): \( g'(a) \) could fail to exist for various reasons, or \( \lim_{x \to a} g'(x) \) could fail to exist, or the two quantities could be unequal.

\( \text{If } 0 < x < 1, \text{ then } g'(x) = -\frac{2}{x^2} + c^2, \text{ which is continuous for each } x \text{ in this interval.} \)

\( \text{If } x > 1, \text{ then } g'(x) = c, \text{ a constant, so is continuous on this interval.} \)

\( \text{But at } x = 1, \text{ we have all of the above reasons to be nervous.} \)

\( \rightarrow \text{ For one, we need } \lim_{x \to 1^+} g'(x) = \lim_{x \to 1^-} g'(x), \text{ i.e. } c = -\frac{2}{1} + c^2. \)

\( \rightarrow \text{ For another, we need } g'(1) \text{ to be defined!} \)

The first requirement means we need \( c^2 - c - 2 = 0 \), so \((c-2)(c+1) = 0\), i.e. \( c = 2 \) or \( c = -1 \). But due to the second requirement, we must reject the case \( c = -1 \), since by part (a), \( g \) is not continuous at \( x = 1 \) in this case, and thus \( g'(1) \) isn't defined! Thus \([c = 2]\) is the only value that produces a function \( g \) where \( g' \) can be continuous everywhere.
3. (15 points) Differentiate, using any method you choose. You do not have to simplify your answers.

(a) \( h(t) = (t - \ln 2t + \cos 3t)^5 \)

\[
h'(t) = 5(t - \ln 2t + \cos 3t)^4 \cdot \left(1 - \frac{1}{2t} \cdot 2 - 3 \sin 3t\right)
\]

(b) \( f(x) = \frac{3x^{\frac{5}{2}}x^2 - 1}{(x + 7)^{10}} \) ← Best to do by log differentiation.

\[
\ln f(x) = \ln(3x^{\frac{5}{2}}x^2) - \ln((x + 7)^{10}) = x \ln 3 + \frac{1}{5} \ln(x^2 - 1) - 10 \ln(x + 7)
\]

\[
\Rightarrow \quad \frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx} \left(x \ln 3 + \frac{1}{5} \ln(x^2 - 1) - 10 \ln(x + 7)\right) = \ln 3 + \frac{1}{5} \cdot \frac{1}{x^2 - 1} \cdot 2x - 10 \cdot \frac{1}{x + 7}
\]

\[
f'(x) = \frac{3x^{\frac{5}{2}}x^2 - 1}{(x + 7)^{10}} \cdot \left(\ln 3 + \frac{2x}{5(x^2 - 1)} - \frac{10}{x + 7}\right)
\]

(c) \( g(z) = \int_{z^2}^{2} \frac{e^t}{\sin t + 2} \, dt \)

Let \( u = z^2 \). Then \( g'(z) = \frac{d}{dz} \int_{z^2}^{2} \frac{e^t}{\sin t + 2} \, dt \)

\[
= \frac{d}{dz} \int_{2}^{z^2} \frac{e^t}{\sin t + 2} \, dt
\]

\[
= \frac{d}{dz} \int_{2}^{u} \frac{e^t}{\sin t + 2} \, dt
\]

(Chain rule, FTC)

\[
= \frac{d}{du} \left(\int_{2}^{u} \frac{e^t}{\sin t + 2} \, dt\right) \cdot \left(\frac{du}{dz}\right) = -\frac{e^u}{\sin u + 2} \cdot 2z = -\frac{e^{z^2}}{\sin z^2 + 2} \cdot 2z
\]
4. (8 points) You are told that a conical funnel has a height of twice its radius, and you wish to calculate its volume. If you measure the radius of the funnel to be 10 cm with a possible error of 0.05 cm, use linear approximations (or differentials) to estimate the maximum error you would obtain in the volume calculation.

Need volume as a function of radius: \[ V = \frac{1}{3} \pi r^2 h \]

\[ \Rightarrow V(r) = \frac{1}{3} \pi r^2 \cdot (2r) = \frac{2}{3} \pi r^3 \]

* Using differentials, error in volume \( \Delta V \) when \( r=10 \) is given by the approximation

\[ \Delta V \approx V'(10) \Delta r \]

where \( V'(r) = 2 \pi r^2 \) and so

\[ V'(10) = 200 \pi \]

Thus

\[ \Delta V \approx V'(10) \Delta r \]

\[ = (200 \pi)(0.05) = \boxed{10 \pi \text{ cm}^3} \]

* Alternative: Using linear approximations, \( V(r) \approx V(10) + V'(10) \cdot (r-10) \) for \( r \) near 10,

so

\[ V(10.05) - V(10) \approx V'(10) \cdot (10.05-10) \]

and

\[ V(9.95) - V(10) \approx V'(10) \cdot (9.95-10) \]

thus, error in \( V \) is approximately \( V'(10) \cdot (0.05) = 10 \pi \text{ cm}^3 \).
5. (8 points) The height of a cylinder is decreasing at a rate of 5 cm/min. If the volume of the cylinder is to be kept constant, at what rate must the radius be increasing when the height is 50 cm and the radius is 80 cm?

We have \( V = \pi r^2 h \), so

\[
\frac{dV}{dt} = \frac{d}{dt}(\pi r^2 h) = \pi \cdot \left( 2r \frac{dr}{dt} \cdot h + r^2 \frac{dh}{dt} \right) = 2\pi rh \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}.
\]

Since the volume is kept constant, we have \( \frac{dV}{dt} = 0 \). Furthermore, when \( h = 50 \) and \( r = 80 \), and \( \frac{dh}{dt} = -5 \), we have

\[
0 = 2\pi rh \frac{dr}{dt} + \pi r^2 \frac{dh}{dt} = 2\pi \cdot 80 \cdot 50 \cdot \frac{dr}{dt} - \pi \cdot 80^2 \cdot 5
\]

\[
\Rightarrow \frac{dr}{dt} = \frac{\pi \cdot 80^2 \cdot 5}{2\pi \cdot 80 \cdot 50} = \frac{80}{20} = \boxed{4} \text{ cm/min}.
\]
6. (10 points) It’s 9:30 p.m. Your snowmobile is out of gas and you are 3 miles due south of a major east-west highway. The nearest service station on the highway is 4 miles east of your position; it closes at midnight. You can walk a mile in 15 minutes on icy roads, but each mile takes 30 minutes on snowy fields.

![Diagram showing the scenario](image)

In order to quickly get to the station, you consider the ways in which you could first walk to some point on the highway and then continue due east the rest of the way.

(a) Determine, with justification, the route that gets you to the station in the least time.

We could of course walk due north to point A labeled above, but this leaves 4 miles of highway. Instead we consider cutting across the snow to end up x miles east of point A, leaving only 4-x miles of highway to walk (see figure).

Then we walk 9+x^2 miles in snow, for \(30 \sqrt{9+x^2}\) minutes, and walk another 4-x miles on the road, for \(15(4-x)\) minutes.

Thus, the time taken in total is: \(Q(x) = 30 \sqrt{9+x^2} + 15(4-x)\); we must minimize \(Q\) for \(0 \leq x \leq 4\). We do this by the Closed Interval Method.

Critical points of \(Q\): \(Q'(x) = 30 \cdot \frac{1}{2} (9+x^2)^{-\frac{1}{2}} (2x) + 15 \cdot (-1) = \frac{30x}{\sqrt{9+x^2}} - 15\); this is never undefined, but \(Q' = 0\) for \(\frac{30x}{\sqrt{9+x^2}} = 15\) \(\Rightarrow 2x = \sqrt{9+x^2} \Rightarrow 4x^2 = x^2 + 9\), so \(x^2 = 3\) and thus \(x = \sqrt{3}\) in our domain.

Testing \(Q\): \(Q(0) = 30 \cdot 3 + 15 \cdot 4 = 150\) min, and \(Q(\sqrt{3}) = 30 \sqrt{12} + 60 - 15 \sqrt{3} = 45 \sqrt{3} + 60\) min, and \(Q(4) = 30 \cdot 5 + 15 \cdot 0 = 150\) min. Since \(Q(\sqrt{3})\) is least (see part (b)), the route should be to go directly across the snow to the point on the road that is \(4 - \sqrt{3}\) miles west of the station.

(b) Can you make it to the service station before it closes? Explain.

Yes: \(Q(\sqrt{3}) = 45 \sqrt{3} + 60 < 45 \cdot 2 + 60 = 150\) because \(\sqrt{3} < 2\), and thus in particular this (quickest) route takes less than 150 min (i.e. 2.5 hrs).
7. (8 points) An airplane takes off at 1:00 p.m. on a 2000 mile flight and arrives at its destination at 5:00 p.m. Use reasoning from calculus to explain why there were at least two times during the journey when the speed of the plane was 400 miles per hour. (You may assume anything you like about the continuity or differentiability of the plane’s position function or one of its derivatives, but state clearly what conditions you are using.)

- Let \( s(t) \) be the position function of the plane, where \( t \) is measured in hours after noon, and to simplify matters the position \( s(t) \) denotes the number of miles from the starting point. (Thus, \( s(1) = 0 \) and \( s(5) = 2000 \).) If we assume that \( s(t) \) is differentiable, then by the Mean Value Theorem, there is some time \( t = c_0 \) such that \( 1 \leq c_0 \leq 5 \) and 
  \[ s'(c_0) = \frac{s(5) - s(1)}{5 - 1} = \frac{2000 \text{ mi}}{4 \text{ hr}} = 500 \text{ mi/hr}. \]
  Since \( s'(t) = v(t) \) is just the velocity function of the plane, this means that there is some time \( t = c_0 \) during the journey when the plane’s velocity is 500 mi/hr.

- Now consider the velocity function \( v(t) \); we have \( v(1) = 0 \), \( v(c_0) = 500 \) for the above time \( c_0 \), and \( v(5) = 0 \). If we assume that \( v(t) \) is continuous, then by the Intermediate Value Theorem, there is some time \( t = c_1 \) with \( 1 \leq c_1 \leq c_0 \) for which \( v(c_1) = 400 \). Also by the IVT, there is some time \( t = c_2 \) with \( c_0 \leq c_2 \leq 5 \) for which \( v(c_2) = 400 \). (In fact \( 1 \leq c_1, c_0 < c_2 \leq 5 \) !)

Thus, there are at least two times (\( c_1 \) and \( c_2 \)) between \( t = 1 \) and \( t = 5 \) when the velocity (hence speed) of the plane is 400 mi/hr.
8. (5 points) Verify by differentiation that \( \int \sec x \, dx = \ln |\sec x + \tan x| + C. \)

We have
\[
\frac{d}{dx} \left( \ln |\sec x + \tan x| \right) = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx} (\sec x + \tan x)
\]

\[
= \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)
\]

\[
= \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x}
\]

\[
= \sec x, \text{ as desired.}
\]
9. (13 points)

(a) Let \( f(x) = 1 + 2x^2 \). Let \( R \) be the region in the \( xy \)-plane bounded by the curve \( y = f(x) \) and the lines \( y = 0 \), \( x = 0 \), and \( x = 3 \). Find the area of \( R \) by evaluating the limit of a Riemann sum that uses the Right Endpoint Rule. (That is, do not use the Fundamental Theorem of Calculus.)

With \( n \) subintervals of equal width, the region \( \{0 \leq x \leq 3\} \) looks like:

\[ \Delta x = \frac{3}{n} \quad \text{and thus} \quad x_i = 0 + i \cdot \Delta x = \frac{3i}{n}. \]

Thus, area of \( R = \int_{0}^{3} (1 + 2x^2) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + 2 \left(\frac{3i}{n}\right)^2\right) \cdot \Delta x \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + 2 \left(\frac{3i}{n}\right)^2\right) \cdot \frac{3}{n} \]

\[ = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left(1 + \frac{54i^2}{n^2}\right) \]

\[ = \lim_{n \to \infty} \left(3 + \frac{54}{n^3} \cdot \sum_{i=1}^{n} i^2 \cdot \frac{1}{n(n+1)(2n+1)}\right) \]

\[ = 3 + \lim_{n \to \infty} \frac{54(n+1)(2n+1)}{6n^3} \]

\[ = 3 + \frac{54}{6} = 3 + \frac{54 \cdot 2}{6} = \boxed{21}. \]
(b) Express the limit
\[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{n} \cos \left( \frac{\pi}{2} + \frac{\pi i}{n} \right) \]
as a definite integral, and evaluate the integral using the Evaluation Theorem.

The sum can be written in long form as
\[ \frac{\pi}{n} \cos \left( \frac{\pi}{2} + \frac{\pi}{n} \right) + \frac{\pi}{n} \cos \left( \frac{\pi}{2} + \frac{2\pi}{n} \right) + \ldots + \frac{\pi}{n} \cos \left( \frac{\pi}{2} + \pi \right) \]
\[ = \frac{\pi}{n} \cdot \left( \cos \left( \frac{\pi}{2} + \frac{\pi}{n} \right) + \cos \left( \frac{\pi}{2} + \frac{2\pi}{n} \right) + \ldots + \cos \left( \frac{\pi}{2} + \pi \right) \right) \]

This suggests that the subinterval width \( \Delta x \) is \( \frac{\pi}{n} \), while the various rectangle heights are \( \cos \left( \frac{\pi}{2} + \frac{\pi}{n} \right) \), \( \cos \left( \frac{\pi}{2} + \frac{2\pi}{n} \right) \), \ldots, etc.

If we let \( f(x) = \cos x \), then these heights are \( f\left( \frac{n}{2} + i \cdot \frac{\pi}{n} \right) = f\left( \frac{\pi}{2} + i \Delta x \right) \) for \( i = 1, \ldots, n \). Thus, \( \frac{b-a}{n} = \frac{\pi}{n} = \Delta x \) and

\[ x_i = a + i \Delta x = \frac{\pi}{2} + i \cdot \frac{\pi}{n} \]so that

\[ b-a = \pi \] and \( a = \frac{\pi}{2} \), meaning \( b = \frac{3\pi}{2} \):

\[ \Delta x = \frac{\pi}{n} \]

Thus,
\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{\pi/2}^{3\pi/2} \cos x \, dx \]
\[ = \sin x \right|_{\pi/2}^{3\pi/2} = \sin \left( \frac{3\pi}{2} \right) - \sin \left( \frac{\pi}{2} \right) = -1 - 1 = -2. \]
10. (10 points) Starting at time $t = 0$ hours, water leaks out of a tank at the rate $r(t)$, measured in gallons per hour. A table of some values for $r(t)$ is given below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(t)$</td>
<td>0</td>
<td>6</td>
<td>11</td>
<td>15</td>
<td>18</td>
<td>20</td>
<td>21</td>
</tr>
</tbody>
</table>

(a) What is the meaning of the quantity $\int_0^2 r(t)dt$? Express your answer in terms relevant to this situation, and make it understandable to someone who does not know any calculus; be sure to use any units that are appropriate.

\[
\int_0^2 r(t)dt \text{ is the net (or total, in this case) amount of water, in gallons, that leaks from the tank between times } t=1 \text{ hr and } t=2 \text{ hrs.}
\]

(b) Use the Midpoint Rule with $n = 3$ to estimate $\int_0^3 r(t)dt$; give your answer as an expression in terms of numbers alone, but you do not have to simplify it.

\[
M_3 = \sum_{i=1}^{3} r\left(\frac{t_{i-1}+t_i}{2}\right) \Delta t = \left(\Delta t \right) \left( r\left(\frac{t_0+t_1}{2}\right) + r\left(\frac{t_1+t_2}{2}\right) + r\left(\frac{t_2+t_3}{2}\right) \right) = \left(\frac{3-0}{3}\right) \left( r\left(\frac{1}{2}\right) + r\left(\frac{3}{2}\right) + r\left(\frac{5}{2}\right) \right) = 6 + 15 + 20 \text{ (gallons)}.
\]

(c) Name another approximation method, including a value of $n$, that can be used to estimate $\int_0^3 r(t)dt$, and write the numerical expression that corresponds to this method and this $n$.

Left Endpoint Rule with $n=6$:

\[
L_6 = \sum_{i=1}^{6} r(t_{i-1}) \Delta t = \left(\frac{3-0}{6}\right) \left( r(0) + r\left(\frac{1}{2}\right) + r\left(\frac{3}{2}\right) + r(2) + r\left(\frac{5}{2}\right) \right) = \frac{1}{2} \left( 0 + 6 + 11 + 15 + 18 + 20 \right) \text{ (gallons)}.
\]

Similarly, Right Endpoint Rule for $n=6$:

\[
R_6 = \frac{1}{2} \left( 6 + 11 + 15 + 18 + 20 + 21 \right) \text{ (gallons)}.
\]

(Other answers that are possible correspond to $n=1, 2, 3, ...$)
11. (22 points) Evaluate each of the following integrals, showing all reasoning.

(a) \[ \int \left( \frac{1}{\sqrt{1-x^2}} - \sec^2 x + \frac{1}{3x} \right) \, dx \]

\[ = \arcsin x - \tan x + \frac{1}{3} \ln |x| + C \]

(b) \[ \int_{-2}^{2} (x - x^3 - \sin x) \, dx = 0 \]

because \( f(x) = x - x^3 - \sin x \) is an odd function:

\[ f(-x) = -x - (-x)^3 - \sin(-x) \]

\[ = -x - (-x^3) + \sin x \]

\[ = -x + x^3 + \sin x = -(x - x^3 - \sin x) = -f(x). \]

(This integral can also be computed directly.)
(c) \[ \int x^2 \left( (1 + x^3)^9 - (1 + x^3)^5 + \cos(1 + x^3) \right) \, dx \]

Let \( u = 1 + x^3 \),

so \( du = 3x^2 \, dx \),

and \( \frac{1}{3} \, du = x^2 \, dx \).

Thus, the integral is

\[
\int \left( u^9 - u^5 + \cos u \right) \cdot \frac{1}{3} \, du = \frac{1}{3} \int \left( u^9 - u^5 + \cos u \right) \, du
\]

\[
= \frac{1}{3} \left( \frac{u^{10}}{10} - \frac{u^6}{6} + \sin u \right) + C
\]

\[
= \frac{1}{3} \left( \frac{(1 + x^3)^{10}}{10} - \frac{(1 + x^3)^6}{6} + \sin (1 + x^3) \right) + C
\]

(d) \[ \int_e^{e^2} \frac{\ln x}{x^3} \, dx \]

Integration by parts: \( u = \ln x \implies du = \frac{1}{x} \, dx \)

\( dv = x^{-3} \, dx \implies v = \frac{x^{-2}}{-2} \).

\[
\int_e^{e^2} \frac{\ln x}{x^3} \, dx = \left[ -\frac{x^{-2}}{2} \ln x \right]_e^{e^2} + \frac{1}{2} \int_e^{e^2} \frac{1}{x^3} \, dx
\]

\[
= \left[ -\frac{x^{-2}}{2} \ln x \right]_e^{e^2} + \frac{1}{2} \int_e^{e^2} \frac{1}{x^3} \, dx
\]

\[
= \left[ -\frac{x^{-2}}{2} \ln x + \frac{1}{2} \cdot \frac{x^2}{e^2} \right]_e^{e^2} = \left( \frac{1}{2} \cdot \frac{1}{e^4} \ln e^2 - \frac{1}{4} \cdot \frac{1}{e^4} \right) - \left( \frac{1}{2} \cdot \frac{1}{e^2} \ln e - \frac{1}{4} \cdot \frac{1}{e^2} \right)
\]

\[
= \left( \frac{1}{e^4} - \frac{1}{e^4} \right) + \frac{1}{2e^2} + \frac{1}{4e^4} = \frac{3}{4e^2} - \frac{5}{4e^4}
\]
(e) $\int \sqrt{x} \cos \sqrt{x} \, dx$

Let $u = \sqrt{x}$, i.e. $u^2 = x$

so $du = \frac{1}{2}x^{-\frac{1}{2}} \, dx$, and $2udu = dx$.

Thus $\int \sqrt{x} \cos \sqrt{x} \, dx = \int u \cdot \cos u \cdot (2udu) = \int 2u^2 \cos u \, du$.

This requires integration by parts: $V = 2u^2$ \quad $dv = 4udu$

$dw = \cos u$ \quad $w = \sin u$

so $\int 2u^2 \cos u \, du = 2u^2 \sin u - \int 4u \sin u \, du$.

The second integral also needs integration by parts: $f = 4u$ \quad $df = 4du$

$dg = \sin u$ \quad $g = -\cos u$

so $\int \sqrt{x} \cos \sqrt{x} \, dx = \int 2u^2 \cos u \, du = 2u^2 \sin u - \int 4u \sin u \, du$

$= 2u^2 \sin u - (\int 4u \cos u - \int -4 \cos u \, du)$

$= 2u^2 \sin u + 4u \cos u - 4 \int \cos u \, du$

$= 2u^2 \sin u + 4u \cos u - 4 \sin u + C$

$= 2x \sin \sqrt{x} + 4x \cos \sqrt{x} - 4 \sin \sqrt{x} + C$
12. (8 points) The figure below shows the graph of a function $f$ that has continuous first, second, and third derivatives. The dashed lines are tangent to the graph of $y = f(x)$ at $(1, 1)$ and $(5, 1)$.

Based on what is shown, determine whether the following integrals are \textit{positive}, \textit{negative}, \textit{zero}, or if there is \textit{not enough information} to tell; give brief explanations.

(a) $\int_1^5 f(x) \, dx$

\textit{Positive}: this integral measures (signed) area under the curve, which lies entirely above $x$-axis over this interval.

(b) $\int_1^5 f'(x) \, dx = f(5) - f(1) = 1 - 1 = 0$.

(c) $\int_1^5 f''(x) \, dx = f''(5) - f''(1) = 0 - (-1) = 1$, so \underline{positive}.

\textit{(using slopes of tangents at 5 & at 1)}

(d) $\int_1^5 f'''(x) \, dx = f'''(5) - f'''(1) = \text{pos - neg}$, so \underline{positive}.

\textit{(using concavity at 5 & at 1)}
13. (17 points) Let \( g(x) = \int_{-4}^{x} f(t) \, dt \), where \( f \) is the function whose graph is given below. Note that the graph of \( f \) is made up of straight lines and a semicircle.

(a) Find each of the following values. If a value is not defined, explain why not.

(i) \( g(3) = \int_{-4}^{3} f(t) \, dt = \frac{1}{2} - \frac{1}{2} \pi (2)^2 + \frac{1}{2} (2)(4) = \frac{7}{2} - 2\pi \)

(ii) \( g'(3) = f(3) = 4 \) (Fundamental Theorem of Calc.)

(iii) \( g''(3) = f''(3) = \text{undefined} \) because \( f \) has a corner at \( x = 3 \).

(b) Identify the critical numbers of \( g \) in the interval \((-5, 5)\).

\( g'(x) = f(x) \) is zero when \( x = -3 \) & \( x = 1 \). \( g'(x) = f(x) \) is not undefined here.

Critical numbers: \([-3, 1]\).

(c) For each critical number that you found in part (b), determine if it is a point where \( g \) has a local maximum, local minimum, or neither. Give reasons for your answer(s).

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sign of ( g'(x) = f(x) )</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3 &lt; x \leq 1)</td>
<td>(-)</td>
<td>( g' ) doesn't change sign at ( x = -3 ), so this is not an extremum.</td>
</tr>
<tr>
<td>( x &gt; 1)</td>
<td>(+)</td>
<td>( g' ) changes from negative to positive at ( x = 1 ), so ( g ) has a local minimum at ( x = 1 ).</td>
</tr>
</tbody>
</table>
For easy reference, here again are the graph of $f$ and the definition of $g$:

\[ g(x) = \int_{-4}^{x} f(t) \, dt \]

(d) On what portion(s) of the interval $(-5, 5)$ is the graph of $g$ concave up? concave down?

\[ g''(x) = f'(x) \] is negative when $f$ is decreasing; i.e., for the intervals $(-3, -1)$ and $(3, 5)$ of the $x$-axis, $g$ is concave down here.

\[ g''(x) = f'(x) \] is positive when $f$ is increasing, i.e., for the intervals $(5, -3)$ and $(-1, 3)$ of the $x$-axis, $g$ is concave up here.

(e) Find a formula that gives the value of $g(x)$ for any $x$ such that $3 \leq x \leq 5$.

**Way #1:** Since $g(x) = f(x) = 7-x$ for $3 \leq x \leq 5$,
we have $g(x) = 7x - \frac{x^2}{2} + C$ on $3 \leq x \leq 5$ for some $C$,
and since $g(3) = \frac{7}{3} - 2\pi$ by part (a), we have

\[ C = g(3) - 7 \cdot 3 + \frac{3^2}{2} = \frac{7}{3} - 2\pi - 21 + \frac{9}{2} = -2\pi - 13, \] so that

\[ g(x) = 7x - \frac{x^2}{2} - 2\pi - 13 \] for $3 \leq x \leq 5$.

**Way #2:** Alternatively, we can think of $g(x) = \int_{-4}^{x} f(t) \, dt = \int_{-4}^{3} f(t) \, dt + \int_{3}^{x} f(t) \, dt$

\[ = (\frac{7}{3} - 2\pi) + \int_{3}^{x} (7-t) \, dt \]

so $g(x) = \frac{7}{3} - 2\pi + \int_{3}^{x} (7-t) \, dt = \frac{7}{3} - 2\pi + \left(7x - \frac{x^2}{2} - \frac{9}{2}\right) = \frac{7x - x^2}{2} - 2\pi - 13$ as before.
14. (8 points) Which of the following shaded areas $A,B,C,D$ are equal? Give reasons. (Hint: you don't need to compute all four areas.)

**Answer:** $A=C=D$, not equal to $B$. [Key: write regions as definite integrals.]

**Reasons:**

$A = \int_0^9 \ln(\sqrt{x}+1) \, dx = \int_1^3 2u \ln(u+1) \, du = D$, so $A=D$; also

$C = \int_1^4 2(x-1) \ln x \, dx = \int_0^3 2u \ln u \, du = C = \int_0^4 2u \ln u - \int_0^4 2u \, du = B - \int_0^4 2u \, du$, so $D=C$ but $C \neq B$. 