Math 41: Calculus
Final Exam — December 8, 2008

Name : ____

Section Leader: (Circle one)	Bob Hough	Joe Rabinoff	David Sher	Nathan Stiennon	Ian Weiner
Section Time: (Circle one)		11:00	1:15		

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers.
- You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- Please check that your copy of this exam contains 19 pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	10		**8**	10	
2	13		**9**	8	
3	10		**10**	13	
4	8		**11**	10	
5	10		**12**	23	
6	10		**13**	12	
7	10		**14**	8	
			Total	155	

1. (10 points) Find each of the following limits, with justification. If there is an infinite limit, then explain whether it is ∞ or $-\infty$.

   (a) $\displaystyle\lim_{x\to 0^+} x^2 e^{(1/x)}$

   (b) $\displaystyle\lim_{x\to\infty} \frac{\ln(1+\ln 4x)}{\ln(2+3\ln x)}$

2. (13 points) Differentiate, using the method of your choice.

(a) $f(x) = x^3 \csc x - \ln|x^2 - 1| + \frac{1}{3}e^{(x^3)}$

(b) $g(t) = \cos\left(t^{1/t}\right)$

(c) $F(z) = \int_0^z \frac{1}{1+t^2}\,dt + \int_0^{1/z} \frac{1}{1+t^2}\,dt$

3. (10 points) Suppose g is the function

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Is g continuous at 0? Use the definition of continuity when justifying your answer.

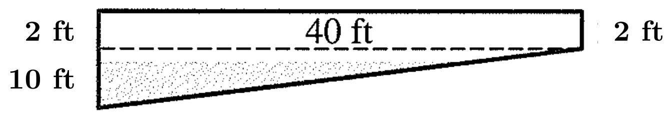
(b) Is g differentiable at 0? Use the limit definition of the derivative when justifying your answer.

4. (8 points)

(a) Find the linearization of the function $f(x) = e^x$ at the point $a = 0$; that is, find the linear function $L(x)$ that best approximates $f(x)$ for values of x near 0.

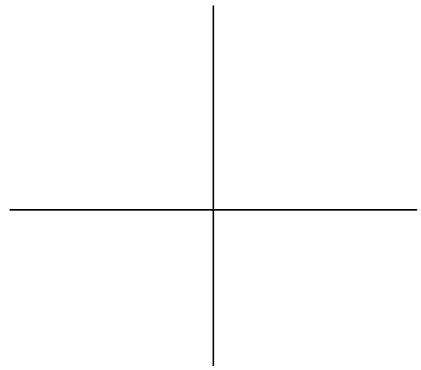
(b) Use the linearization to estimate $e^{-0.15}$. Is your approximation an overestimate or underestimate of the actual value? Explain fully.

5. (10 points) A swimming pool is 40 feet long and 20 feet wide. Its depth varies uniformly from 2 feet at the shallow end to 12 feet at the deep end. (The figure below is a side view, where the length and depths are shown.) Suppose that the pool is being filled at the rate of 150 ft^3/min. At what rate is the depth of water at the deep end rising when the depth there is 6 feet?



6. (10 points) Let $h(x) = \left|x^2 - 3\right|$.

(a) On the axes below, sketch the graph of $y = x^2 - 3$, and use this to sketch $y = h(x)$.



(b) State the Mean Value Theorem.

(c) Find the slope of the secant line between the points $(1, h(1))$ and $(2, h(2))$.

(d) Find a number c such that the tangent line to $y = h(x)$ at $x = c$ is parallel to this secant line.

(e) There is no such number c between 1 and 2. Does this contradict the Mean Value Theorem? Explain.

7. (10 points) An eccentric local cheesemaker has 1,000 meters of fencing to enclose two *square* pastures: one for goats and one for sheep. The two pastures must be kept far apart from one another, so no amount of fencing may be shared for the two squares. Although the eventual milk yield from the animals will someday go to new cheesemaking ventures, he will see sizable startup costs, as follows:

   - each square-meter of goat pasture costs him 3 dollars, and
   - each square-meter of sheep pasture costs him 2 dollars.

   Finally, to comply with a new state law, each square pasture can be no smaller than 50-meters-by-50-meters. If he wants to use all the available fencing, what should be the size of each square pasture so that his total startup costs are minimized? Justify your answer and show all steps in your reasoning.

8. (10 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary. All instances of a function f refer to a function that is continuous on its domain.

T **F** The absolute minimum value of $1 - x^2$ on $[-1, 2]$ is -3.

T **F** If f has a local minimum at c and f is differentiable at c, then $f''(c) > 0$.

T **F** An inflection point of f always occurs at a critical number of f.

T **F** If f has an absolute maximum at c and f is differentiable at c, then $f'(c) = 0$.

T **F** If $f'(c) = 0$, then f has a local extremum at c.

T **F** If c is a critical number of f, then $f'(c) = 0$.

T **F** If $f''(x) > 0$ at all points on the x-axis and $f'(c) = 0$, then f has its absolute minimum at c.

T **F** If $f'(x) > 0$ on $[a, b]$, then f is increasing on $[a, b]$.

T **F** If $f''(x) > 0$ on $[a, b]$, then f is concave up on $[a, b]$.

T **F** If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

9. (8 points)

(a) Verify the following indefinite integral expression by differentiating.

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x+\sqrt{x^2-1}\right| + C$$

(b) What is wrong with the equation below? Be precise in your answer.

$$\int_{-1}^{1} \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x+\sqrt{x^2-1}\right|\Big]_{-1}^{1} = \ln 1 - \ln|-1| = 0$$

10. (13 points)

(a) Let $f(x) = 5x + x^3$. Let R be the region in the xy-plane bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = 0$, and $x = 2$. Find the area of R by evaluating the limit of a Riemann sum that uses the *Right Endpoint Rule*. (That is, do not use the Fundamental Theorem of Calculus.)

(b) Express the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \sqrt{9 - \left(-3 + \frac{6i}{n}\right)^2}$$

as a definite integral, and evaluate the integral by interpreting it as an area.

11. (10 points) Suppose $E(t)$ is the US carbon emissions rate, in millions of tons per year, at the year t, where t is measured in years since 1940. A table of some values for $E(t)$ is given below.

t	0	10	20	30	40	50
$E(t)$	6.9	9.4	13.0	18.5	20.9	19.6

(a) What does the quantity $\int_{10}^{68} E(t)dt$ represent? Express your answer in terms relevant to this situation, and make it understandable to someone who does not know any calculus; be sure to use any units that are appropriate.

(b) Use the *Midpoint Rule* with $n = 2$ to estimate $\int_{0}^{40} E(t)dt$; give your answer as an expression in terms of numbers alone, but you do not have to simplify it.

(c) Use the *Left Endpoint Rule* with $n = 4$ to estimate $\int_{20}^{60} E(t)dt$; again give your answer in terms of numbers alone.

12. (23 points) Evaluate each of the following integrals, showing all reasoning.

(a) $\displaystyle\int\left(3(1-x^2)^{-1/2}+\frac{x^2-2x}{\sqrt{x}}-\frac{1}{e}\right)\,dx$

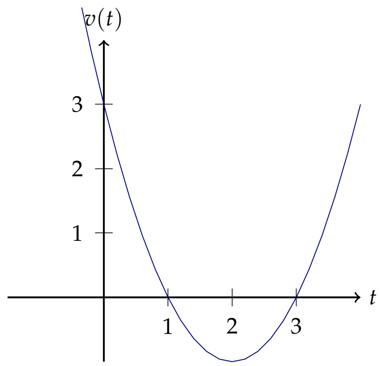
(b) $\displaystyle\int_{-1}^{1}|x|\,dx$

(c) $\int \tan x \, dx$

(d) $\int_{e}^{e^3} \frac{\ln x}{x^2} \, dx$

(e) $\int \arctan x \, dx$

13. (12 points) Suppose you are standing alongside a road. At time $t = 0$ minutes, a car passes by you traveling forward in the positive x-direction. The car's *velocity* $v(t)$ (in km per minute) is given by the graph below:



Use the diagram to answer the following questions. *Give reasons for your answers.*

(a) Let $s(t)$ be the car's position function. Place the numbers $s(1)$, $s(2)$, $s(3)$ in ascending order (that is, from smallest to largest).

(b) For each of the times $t = 1, 2, 3$, is the car moving forward, backward, or neither? At each of these times, is its acceleration positive, negative, or zero?

(c) Given that $\int_0^1 v(t)dt = 1.3$ and $\int_1^3 v(t)dt = -1.5$, find the total distance traveled by the car between $t = 0$ and $t = 3$. And if you have stayed in place throughout this interval, where would you find the car located at $t = 3$?

14. (8 points)

(a) Find a function f and a positive constant a such that $\displaystyle\int_a^{x^2} f(t)\,dt = \frac{x^3}{3} - 1$ for all positive x.

(b) Does there exist a function g such that $\displaystyle\int_0^{x} g(t)\,dt = x + 1$ for all x? If so, find such a g; if not, explain why.

Formulas for Reference

Geometric Formulas:

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3, \qquad SA_{\text{sphere}} = 4\pi r^2, \qquad A_{\text{circle}} = \pi r^2$$

$$V_{\text{cylinder}} = \pi r^2 h, \qquad V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

Summation Formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$