Math 41: Calculus
Final Exam — December 7, 2009

Name: ____ **SUID#:** ______________

Select your section:				
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02 (11-11:50am)	08 (10-10:50am)	04 (1:15-2:05pm)	03 (11-11:50am)	41A
07 (10-10:50am)	10 (11-11:50am)	09 (2:15-3:05pm)	06 (1:15-2:05pm)	

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers.
- You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- Please check that your copy of this exam contains 18 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	9		**8**	5	
2	14		**9**	9	
3	8		**10**	14	
4	10		**11**	10	
5	18		**12**	14	
6	10		**13**	23	
7	6		**14**	10	
			Total	160	

1. (9 points) Find each of the following limits, with justification. If there is an infinite limit, then explain whether it is ∞ or $-\infty$.

   (a) $\displaystyle\lim_{x\to-\frac{\pi}{2}^+} \frac{x+\frac{\pi}{2}}{\sin x + 1}$

   (b) $\displaystyle\lim_{x\to 0^+} x^{(e^x-1)}$

2. (14 points) In each part below, use the method of your choice to find the derivative.

   (a) Find $h'(z)$ if $h(z) = \sqrt{z\cos(z^3)} + \csc^2 z$.

   (b) Find $\frac{dy}{dx}$ if $x^y = y^x$.

   (c) Find $p'(x)$ if $p(x) = \displaystyle\int_x^{2x} \sqrt{1+t^2}\,dt$.

3. (8 points) Suppose all we are given about the function $g(x)$ is that

$$g(2) = 7 \quad \text{and} \quad g'(x) = \sqrt{1 + x^3} \text{ for all } x.$$

(a) Use a linear approximation to estimate $g(2.01)$ and $g(1.98)$; show all the steps of your reasoning.

(b) Are your estimates in part (a) too large or too small? Explain.

4. (10 points) A child's kite is maintaining a constant height of 100 feet above the ground. There is a strong wind blowing the kite away from the child, in such a way that the kite is moving *horizontally* at a speed of 7 ft/sec.

   (a) At what rate is the child releasing string when 300 feet of string have already been released?

   (b) At what rate is the angle between the string and the horizontal decreasing at this same moment? (Assume the child has negligible height herself.)

5. (18 points) Let $f(x) = x^4 - 3x + 1$. For this problem, we will investigate certain positive solutions to the equation $f(x) = 0$. (Note: you don't have to solve the earlier parts of this problem to solve the subsequent ones; just cite any part's stated result if you need it.)

   (a) Show that $x^4 - 3x + 1 = 0$ for at least one x-value in the interval $[1, 2]$. Explain your reasoning completely; however, you don't have to find the exact value.

   (b) Let $h(x)$ be a differentiable function, and suppose that $h(a) = h(b) = 0$ for two different values a and b. Show that there exists a value c between a and b with $h'(c) = 0$.

   (c) Now show that $f(x) = x^4 - 3x + 1 = 0$ for *exactly* one x-value in $[1, 2]$. (Hint: find f' and use the result of part (b) to explain why there can't be more than one value in this interval.)

(d) Based on part (c), the equation $x^4 - 3x + 1 = 0$ has exactly one solution x in $[1, 2]$. Use Newton's method with initial guess $x_1 = 1$ to produce two successive approximations to the solution, namely x_2 and x_3. Show all of your steps, and simplify your answers as much as you can.

6. (10 points) A *silo* consists of a cylindrical tower of height h and radius r, capped by a hemisphere of radius r. (There is no material used for the bottom.) Thus, a formula for its surface area is

$$A_{\text{silo}} = 2\pi rh + 2\pi r^2.$$

Suppose there is enough material available to build a silo with surface area equal to 500 square feet. What is the maximum possible volume of such a silo? Justify your answer. (Note: if you use the volume formulas on the reference sheet, remember that a hemisphere is *half* a sphere.)

7. (6 points) Mark each statement below as *true* or *false* by circling TRUE or FALSE. *No justification is necessary.*

The statement "$\lim_{x\to 2}(-10+6x) = 2$" can be rigorously justified by saying that for any positive ϵ, we have TRUE FALSE

$$|(-10+6x)-2| < \epsilon \quad \text{whenever} \quad |x-2| < \epsilon/2.$$

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$$|(-10+6x)-2| < \epsilon \quad \text{whenever} \quad |x-2| < \epsilon/60.$$

If the differentiable functions $f(x)$ and $g(x)$ are positive and increasing, then the product $f(x)g(x)$ must also be positive and increasing. TRUE FALSE

If the differentiable functions $f(x)$ and $g(x)$ are positive and concave up, then the product $f(x)g(x)$ must also be concave up. TRUE FALSE

If the differentiable function $f(x)$ is odd, then $f'(x)$ must be even. TRUE FALSE

8. (5 points) Verify the following indefinite integral expression by differentiating, showing your steps. (Here a and b are constants.)

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$$

9. (9 points) The function $f(x)$ is differentiable. In the table below are some values of $f(x)$ and its derivative $f'(x)$.

x	1	2	3	4	5	6
$f(x)$	3	2	0	1	4	5
$f'(x)$	-3	-2	-1	2	3	4

(a) Approximate $\int_0^6 f(x)\,dx$ by a Riemann sum using three rectangles, with midpoints as sample points.

(b) Approximate $\int_0^6 f(x)\,dx$ by a Riemann sum using six rectangles, with right endpoints as sample points.

(c) Find the exact value of $\int_1^4 f'(x)\,dx$. Show your steps.

10. (14 points)

(a) Let $f(x) = x^2$. Let R be the region in the xy-plane bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = 1$, and $x = 4$. Find the area of R by evaluating the limit of a Riemann sum that uses the *Right Endpoint Rule*. (That is, do not use the Fundamental Theorem of Calculus.)

(b) Show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{2n} \tan\left(-\frac{\pi}{4} + \frac{i\pi}{2n}\right) = 0$$

(Hint: first show how to express the limit as a definite integral, then justify the value of the integral.)

11. (10 points) A hybrid vehicle is traveling north from Los Angeles on Interstate 5; suppose we measure its position by the number of miles north of LA it is located. Let $f(z)$ be the rate, in gallons per mile, at which the vehicle is consuming fuel with respect to position when the vehicle is located z miles north of Los Angeles.

   (a) What does the quantity $\int_{30}^{50} f(z)\,dz$ represent? Express your answer in terms relevant to this situation, and make it understandable to someone who does not know any calculus; be sure to use any units that are appropriate.

   (b) Let $s(t)$ be the position, in miles north of Los Angeles, of the vehicle at time t (in hours), and suppose $s(0) = 0$. Define the function

$$H(t) = \int_0^{s(t)} f(z)\,dz.$$

   What does $H(t)$ represent? Again use simple terminology relevant to this situation, and any appropriate units.

   (c) Find an expression for $H'(t)$ in terms of the functions f and s (and possibly their derivatives). What are the units of $H'(t)$?

12. (14 points) Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Finally, let

$$g(x) = \int_0^x f(t)\,dt.$$

Each of the statements about g below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which. You must give a brief justification in order to receive credit.

g is a differentiable function of x. T F MAYBE

g is a continuous function of x. T F MAYBE

The graph of $g(x)$ has a horizontal tangent at $x = 1$. T F MAYBE

g has a local maximum at $x = 1$. T F MAYBE

g has a local minimum at $x = 1$. T F MAYBE

The graph of g has a point of inflection at $x = 1$. T F MAYBE

The graph of $\frac{dg}{dx}$ crosses the x-axis at $x = 1$. T F MAYBE

13. (23 points) Show all reasoning when solving each of the problems below.

(a) Find a formula for $f(x)$ if f is continuous, $f(0) = 1$, and

$$f'(x) = \begin{cases} 3 & \text{if } x < 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$$

(b) $\displaystyle\int \left(\frac{1 + 2z + 3z^2}{4z^2} - \frac{\pi}{1 + z^2} + \sec z \tan z \right) dz$

(c) $\displaystyle\int_{e}^{e^3} \frac{dx}{x\sqrt{\ln x}}$

(d) $\displaystyle\int t^2 e^{-t}\, dt$

(e) $\int \sin(\sqrt{x})\, dx$

14. (10 points)

(a) Show that $0 \leq \int_0^1 x^4 \sin x \, dx \leq \frac{1}{5}$.

(b) Suppose $f'(x) = \frac{\cos x}{x}$, and let $f\left(\frac{\pi}{2}\right) = a$ and $f\left(\frac{3\pi}{2}\right) = b$ for constants a and b. Find the value of the integral

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x) \, dx$$

Your answer will involve a and b.

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Formulas for Reference

Geometric Formulas:

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3, \qquad SA_{\text{sphere}} = 4\pi r^2, \qquad A_{\text{circle}} = \pi r^2$$

$$V_{\text{cylinder}} = \pi r^2 h, \qquad V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

Summation Formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Quick Survey (anonymous)

In order for us to better understand how you used your Stewart textbook this quarter, could you please answer the following two questions:

1. When you referred to the textbook this quarter, such as for homework problems or readings, what form did the book take? **Please circle one:**
   - Almost always used a hardbound copy of the textbook (my own, a friend's, the library's, etc.)
   - Almost always used an "electronic" copy (or printouts of one)
   - I used a mix of both forms.

2. ***If you had it to do over again,*** knowing how you used the Stewart text in this course and all the issues of price, would you find it acceptable just to access an electronic version of this textbook, or would you prefer to use a hardbound copy? **Please circle one:**
   - I'd just use an electronic copy of Stewart exclusively.
   - I'd prefer to use a hardbound version for at least some portion of the time.
   - No preference.

When you've answered, please quickly pass this page forward to the section leaders. Thanks!