1. (13 points) Differentiate, using the method of your choice.

(a) \( p(t) = \ln(\sec t + \tan t) + \log_2(2 + t) \)

Using the rule for the derivative of a sum,

\[
p'(t) = \frac{d}{dt} (\ln(\sec t + \tan t) + \log_2(2 + t)) = \frac{d}{dt} \ln(\sec t + \tan t) + \frac{d}{dt} \log_2(2 + t)
\]

Now, we do each separate derivative using the Chain Rule:

\[
\frac{d}{dt} \ln(\sec t + \tan t) = \frac{1}{\sec t + \tan t} \frac{d}{dt} (\sec t + \tan t) = \sec t \tan t + \sec^2 t
\]

and

\[
\frac{d}{dt} \log_2(2 + t) = \frac{1}{\ln 2} \frac{d}{dt} \ln(2 + t) = \frac{1}{(\ln 2)(2 + t)} \frac{d}{dt} (2 + t) = \frac{1}{(\ln 2)(2 + t)}
\]

Combining, we get that

\[
p'(t) = \sec t + \frac{1}{(\ln 2)(2 + t)}
\]

(b) \( f(x) = 10^{\sin(\pi x)} + \sqrt{x + \sqrt{x}} \)

Again, we decompose using the rule for a sum:

\[
f'(x) = \frac{d}{dx} 10^{\sin(\pi x)} + \frac{d}{dx} \sqrt{x + \sqrt{x}}
\]

For the first term, using the chain rule and the fact that \( \frac{d}{dx} (a^x) = \ln a \cdot a^x \), we get

\[
\frac{d}{dx} 10^{\sin(\pi x)} = \ln 10 \cdot 10^{\sin(\pi x)} \frac{d}{dx} \sin(\pi x) = \ln 10 \cdot \pi \cdot 10^{\sin(\pi x)} \cos(\pi x)
\]

Furthermore, using the chain rule for the second term,

\[
\frac{d}{dx} \sqrt{x + \sqrt{x}} = \frac{1}{2} \left(x + x^{1/2}\right)^{-1/2} \frac{d}{dx} \left(x + x^{1/2}\right) = \frac{1}{2} \left(x + x^{1/2}\right)^{-1/2} \left(1 + \frac{1}{2} x^{-1/2}\right) = \frac{1}{2(\sqrt{x + \sqrt{x}})} \left(1 + \frac{1}{2\sqrt{x}}\right)
\]

Combining the above, we get that

\[
f'(x) = \ln 10 \cdot \pi \cdot 10^{\sin(\pi x)} \cos(\pi x) + \frac{1}{2(\sqrt{x + \sqrt{x}})} \left(1 + \frac{1}{2\sqrt{x}}\right)
\]
(c) \( g(x) = \arctan(x^{\cos x}) \)

(5 points) Again using the chain rule,

\[
\frac{d}{dx} \arctan(x^{\cos x}) = \frac{1}{1 + (x^{\cos x})^2} \frac{d}{dx} (x^{\cos x})
\]

\[
= \frac{1}{1 + x^{2 \cos x}} \frac{d}{dx} (x^{\cos x})
\]

Now we need to calculate \( \frac{d}{dx} (x^{\cos x}) \). Note that we can’t use the rule that \( \frac{d}{dx} x^n = nx^{n-1} \), which only works when \( n \) is constant; and we can’t use the rule that \( \frac{d}{dx} a^x = \ln a \cdot a^x \), which only works when \( a \) is constant. Thus, we do the following:

\[
\frac{d}{dx} (x^{\cos x}) = \frac{d}{dx} (e^{\ln(x^{\cos x})}) = \frac{d}{dx} (e^{\cos x \ln x})
\]

\[
= e^{\cos x \ln x} \frac{d}{dx} (\cos x \ln x)
\]

\[
= e^{\cos x \ln x} \left( \frac{\cos x}{x} - \sin x \ln x \right)
\]

\[
= x^{\cos x} \left( \frac{\cos x}{x} - \sin x \ln x \right)
\]

(We could have also set \( y = x^{\cos x} \), calculated that \( \ln y = \cos x \ln x \) and used implicit differentiation.) Combining, we find

\[
g'(x) = \frac{x^{\cos x}}{1 + x^{2 \cos x}} \left( \frac{\cos x}{x} - \sin x \ln x \right)
\]
2. (12 points) Consider the curve with equation \( x^2y^2 + xy = 2 \).

(a) Find an expression for \( \frac{dy}{dx} \). (Your answer can be in terms of both \( x \) and \( y \).)

(4 points) We differentiate both sides with respect to \( x \):

\[
\frac{d}{dx} (x^2y^2 + xy) = \frac{d}{dx} (2)
\]

\[
\implies 2xy^2 + 2x^2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0
\]

\[
\implies x(2xy + 1)\frac{dy}{dx} = -y(2xy + 1)
\]

So solving for \( \frac{dy}{dx} \), we get

\[
\frac{dy}{dx} = -\frac{y}{x}
\]

(b) There is a unique point on the curve with \( x = -1 \) and positive \( y \); find this point and write the equation of the tangent line to the curve at this point.

(4 points) Substituting \( x = -1 \) into \( x^2y^2 + xy = 2 \), we get

\[
y^2 - y = 2
\]

\[
\implies (y - 2)(y + 1) = 0
\]

So there are two points with \( x \)-coordinate equal to \(-1\) on the curve. The only one with positive \( y \)-coordinate is \( (x, y) = (-1, 2) \). The slope of the curve at this point is given by

\[
m = \frac{-2}{-1} = 2
\]

So the equation of the tangent line is

\[
y - 2 = 2(x + 1)
\]
For easy reference, the curve has equation $x^2y^2 + xy = 2$.

(c) Find all points on the curve where the slope of the tangent line is $-1$.

(4 points) The slope condition implies $-\frac{y}{x} = -1$; hence the points must satisfy $y = x$. Substituting back to the original equation, one gets

$$x^4 + x^2 = 2 \Rightarrow (x^2 - 1)(x^2 + 2) = 0$$

So the only real solutions are $x = 1$ and $x = -1$. Now using the condition $y = x$ again, we find that $(x, y) = (1,1)$ or $(-1, -1)$ are the only points satisfying the requirement of (c).
3. (9 points)

(a) Use linear approximation to estimate \( e^{1.1} \), showing your reasoning. Express your answer as a simplified rational multiple of \( e \); that is, give your answer in the form \( \frac{a}{b} e \), where \( a \) and \( b \) are whole numbers.

(6 points) We use linear approximation to estimate the value \( e^{1.1} \). Indeed, defining the function

\[ f(x) = \frac{e^x}{\sqrt{x}} \]

then the value we’re approximating is

\[ f(1.1) = f(1 + 0.1) = f(a + \Delta x) \],

where \( a = 1 \) and \( \Delta x = 0.1 \). We need to compute \( f'(1) \). Differentiating, we find

\[ f'(x) = \frac{e^x \cdot \sqrt{x} - e^x / (2\sqrt{x})}{(\sqrt{x})^2} = e^x \left( x^{1/2} - \frac{1}{2} x^{-3/2} \right) \].

Substituting in \( x = 1 \), we obtain

\[ f'(1) = \frac{e^{1/\sqrt{1}} - e^{1/(2\sqrt{1})}}{(\sqrt{1})^2} = \frac{e}{2} \].

We can now proceed by either taking the linearization of \( f \) or estimating using differentials, observing that \( f(a) = f(1) = e \). Using the first method, we obtain the linearization \( L(x) \) of \( f \); that is, the line of slope \( f'(a) = \frac{e}{2} \) passing through the point \( (a, f(a)) = (1, e) \):

\[ L(x) = f'(a)(x - a) + f(a) = \frac{e}{2} (x - 1) + e. \]

Hence, we get

\[ L(1.1) = \frac{e}{2} (1.1 - 1) + e = \frac{e}{2} \cdot \frac{1}{10} + e = \left( \frac{21}{20} \right) e. \]

Alternatively, we can use differentials:

\[ f(1.1) = f(a + \Delta x) = f(a) + \Delta y \]

\[ \approx f(a) + \frac{dy}{dx} \cdot \Delta x = e + \frac{e}{2} \cdot \frac{1}{10} = \left( \frac{21}{20} \right) e. \]

(b) Is your estimate in part (a) larger or smaller than the actual value? Justify your answer.

(3 points) We calculate the second derivative of \( f \):

\[ f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} e^x \left( x^{-1/2} - \frac{1}{2} x^{-3/2} \right) = e^x \left( x^{-1/2} - \frac{1}{2} x^{-3/2} \right) + e^x \left( -\frac{1}{2} x^{-3/2} + \frac{3}{4} x^{-5/2} \right) \].

Hence,

\[ f''(1) = \frac{3e}{4} > 0, \]

so \( f \) opens concave-upwards; thus the tangent line \( y = L(x) \) will lie below the curve \( y = f(x) \), and so the above estimate is an under-approximation.
4. (10 points) A caffeinated beverage is brewed by placing water in an inverted cone-shaped filter of radius 3 cm and height 5 cm, and allowing it to leak through the cone’s vertex into a cylindrical cup of the same radius and height. (You may assume the cone is suspended well above the top of the cup.) The leaking is such that the water level in the cone filter decreases at a rate of 0.5 cm/sec. How fast is the water level in the cup rising when the water level in the cone filter is 2 cm above the vertex?

Formulas for reference: the volumes of a circular cone and a circular cylinder, each of height $h$ and base radius $r$, are given, respectively, by

\[ V_{\text{cone}} = \frac{1}{3} \pi r^2 h, \quad V_{\text{cyl}} = \pi r^2 h. \]

Let $V_1$ be the volume of water in the cone and $V_2$ the volume of water in the cylinder. Let $h$ be the water level in the cone and $r$ the radius of the surface of water in the cone. Finally, let $y$ be the water level in the cylinder. We’re given that $\frac{dh}{dt} = -0.5$ cm/sec, and we need to find $\frac{dy}{dt}$ when $h = 2$ cm.

Let us find the volume of water inside the cone as a function of $h$. Using similar triangles, we see that $r = \frac{3h}{5}$, and plugging this back in the formula for the volume of a cone we get

\[ V_1 = \frac{\pi}{3} \left( \frac{3h}{5} \right)^2 h = \frac{\pi 9h^2}{3 \cdot 25} h = \frac{3\pi}{25} h^3. \]

The radius of the surface of the water in the cylinder is constant and equal to 3 cm, so $V_2 = 9\pi y$. Taking the derivatives of these volumes with respect to $t$, we find

\[ \frac{dV_1}{dt} = \frac{9\pi}{25} h^2 \frac{dh}{dt} \quad \text{and} \quad \frac{dV_2}{dt} = 9\pi \frac{dy}{dt}. \]

Since the water leaking from the cone is falling into the cylinder, we have

\[ V_1 + V_2 = \text{constant} \quad \Rightarrow \quad \frac{dV_2}{dt} = -\frac{dV_1}{dt}. \]

Now, plugging $h = 2$ and $\frac{dh}{dt} = -0.5$, we get:

\[ \frac{dV_2}{dt} = -\frac{dV_1}{dt} = -\left( \frac{9\pi}{25} \cdot 4 \cdot (-0.5) \right) = \frac{18\pi}{25} \text{ cm}^3/\text{sec} \quad \Rightarrow \quad \frac{18\pi}{25} = 9\pi \frac{dy}{dt} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{2}{25} \text{ cm/sec}. \]

Therefore, the water level in the cup is rising at a rate of $\frac{2}{25}$ cm/sec.
5. (13 points) Find each of the following limits, with justification. If there is an infinite limit, then explain whether it is \( \infty \) or \( -\infty \).

(a) \( \lim_{x \to 4} \frac{f(x) - 1}{x - 4} \), where \( f \) is such that the tangent line to \( f \) at \( x = 4 \) has equation \( 2x - 4y = 4 \).

(You can assume that \( f \) and all of its derivatives are continuous near \( x = 4 \).)

(5 points) The tangent line to the graph of \( y = f(x) \) at \( x = 4 \) passes through the point \( (4, f(4)) \). Therefore we can find \( f(4) \) from the equation of the tangent line:

\[
2 \cdot 4 - 4 \cdot f(4) = 4 \implies f(4) = 1
\]

Then the given limit turns out to be indeterminate of the form \( \frac{0}{0} \), and since both functions are differentiable, we can use L'Hôpital's rule:

\[
\lim_{x \to 4} \frac{f(x) - 1}{x - 4} = \lim_{x \to 4} \frac{f'(x)}{1} = \lim_{x \to 4} f'(x) = f'(4)
\]

where the last equality follows because we'll assume \( f'(x) \) is continuous. So we only need to find \( f'(4) \). Remember that \( f'(4) \) is the slope of the tangent line at \( x = 4 \), and this is \( 2x - 4y = 4 \), which can be rewritten as \( y = \frac{2x - 4}{4} = \frac{1}{2}x - 1 \). Therefore, \( f'(4) = \frac{1}{2} \). Hence:

\[
\lim_{x \to 4} \frac{f(x) - 1}{x - 4} = \frac{1}{2}
\]

**Alternate solution:** Define \( g(x) = f(x) - 1 \) and \( h(x) = x - 4 \), so that the limit we seek is

\[
L = \lim_{x \to 4} \frac{f(x) - 1}{x - 4} = \lim_{x \to 4} \frac{g(x)}{h(x)}.
\]

Observe that the tangent line to \( g(x) \) at \( x = 4 \) has equation \( 2x - 4y = 8 \); to see this, notice that \( f(x) \) passes through the point \((0, -1)\), whereas \( g(x) \) passes through the point \((0, -2)\). Hence,

\[
\lim_{x \to 4} g(x) = g(4) = 0 \quad \text{and} \quad g'(4) = 1/2.
\]

It is easily verified that

\[
\lim_{x \to 4} h(x) = h(4) = 0 \quad \text{and} \quad h'(4) = 1.
\]

Hence, by l'Hôpital's rule applied to the indeterminate form \( 0/0 \),

\[
L = \frac{g'(4)}{h'(4)} = \frac{1/2}{1} = \frac{1}{2}.
\]
(b) \( \lim_{x \to \infty} (x - \ln x) \)

(4 points) This is an indeterminate of the form \( \infty - \infty \); if we try to “factor” this expression to write it as a product, we find:

\[
\lim_{x \to \infty} (x - \ln x) = \lim_{x \to \infty} x \left( 1 - \frac{\ln x}{x} \right)
\]

and we observe that \( \lim_{x \to \infty} \frac{\ln x}{x} \) is indeterminate of the form \( \infty/\infty \). Thus, by l'Hôpital’s Rule,

\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0,
\]

so that

\[
\lim_{x \to \infty} (x - \ln x) = \lim_{x \to \infty} x \left( 1 - \frac{\ln x}{x} \right) = \infty
\]

because the right factor in the last expression approaches 1 while the left factor approaches \( \infty \).

Alternate solution: Using the fact that \( \ln(e^t) = t \), we can write

\[
\lim_{x \to \infty} (x - \ln x) = \lim_{x \to \infty} \left[ \ln e^{(x-\ln x)} \right].
\]

By continuity of natural log, we can first analyze the behavior of the inner function \( u = e^{(x-\ln x)} \):

\[
\lim_{x \to \infty} e^{x-\ln x} = \lim_{x \to \infty} \frac{e^x}{x} \quad \text{(an } \infty/\infty \text{ indeterminate, since each factor approaches } \infty) \\
= \lim_{x \to \infty} \frac{e^x}{1} \quad \text{(by l'Hôpital’s rule)} \\
= \infty.
\]

Thus,

\[
\lim_{x \to \infty} (x - \ln x) = \lim_{x \to \infty} \ln e^{(x-\ln x)} = \lim_{u \to \infty} \ln u = \infty.
\]

(c) \( \lim_{x \to 0^+} (\cos x)^{1/\sin^2 x} \)

(4 points) Let \( \exp(x) = e^x \). Using the fact that \( \exp(\ln x) = x \), we obtain

\[
\lim_{x \to 0^+} \cos(x)^{1/\sin^2(x)} = \lim_{x \to 0^+} \left[ \exp \ln \cos(x)^{1/\sin^2(x)} \right]
\]

\[
= \exp \left[ \lim_{x \to 0^+} \ln \cos(x)^{1/\sin^2(x)} \right] \quad \text{(by continuity of exp)}
\]

\[
= \exp \left[ \lim_{x \to 0^+} \frac{\ln \cos(x)}{\sin^2(x)} \right] \quad \text{(a } 0/0 \text{ indeterminate form)}
\]

\[
= \exp \left[ \lim_{x \to 0^+} \frac{-\tan x}{2\sin x \cos x} \right] \quad \text{(by l'Hôpital’s rule)}
\]

\[
= \exp \left[ \lim_{x \to 0^+} \frac{-1}{2 \cos^2 x} \right] \quad \text{(tan } x = \frac{\sin x}{\cos x})
\]

\[
= \exp \left[ -\frac{1}{2} \right] = e^{-1/2}.
\]

The rule of l'Hôpital was applied to the indeterminate form \( 0/0 \), since

\[
\lim_{x \to 0^+} \ln x = \ln 1 = 0 \quad \text{and} \quad \lim_{x \to 0^+} \sin^2 x = 0.
\]
6. (10 points) Let \( f(x) = 9x^{2/3} - x^{5/3} \).

(a) Find the critical numbers of \( f \), and characterize each as a point of local minimum, local maximum, or neither. Show all steps of your reasoning.

(7 points) A value \( x = c \) in the domain of \( f \) is a critical number if and only if either \( f'(c) = 0 \) or \( f' \) is undefined at \( c \); we first compute \( f'(x) \) and simplify:

\[
f'(x) = 9 \cdot \frac{2}{3} x^{-1/3} - \frac{5}{3} x^{2/3} = \frac{6}{x^{1/3}} - \frac{5x^{2/3}}{3} = \frac{18 - 5x}{3x^{1/3}}.
\]

Clearly \( f'(x) = 0 \) if and only if \( 18 - 5x = 0 \), and \( f' \) is undefined if and only if \( x^{1/3} = 0 \). Thus, there are two critical numbers, \( x = \frac{18}{5} \) and \( x = 0 \).

To determine the nature of each critical number, we next note that \( 18 - 5x > 0 \) if and only if \( x < \frac{18}{5} \), and that \( x^{1/3} > 0 \) if and only if \( x > 0 \). This leads to the following chart:

<table>
<thead>
<tr>
<th>interval</th>
<th>sign of ( 18 - 5x )</th>
<th>sign of ( x^{1/3} )</th>
<th>sign of ( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; 0 )</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( 0 &lt; x &lt; \frac{18}{5} )</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x &gt; \frac{18}{5} )</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

We’ve found that at \( x = 0 \), \( f' \) is changing from negative to positive; and at \( x = \frac{18}{5} \), \( f' \) is changing from positive to negative. Thus, by the First Derivative Test, \( f \) has a local minimum at \( x = 0 \), and a local maximum at \( x = \frac{18}{5} \).

**Note:** We could instead use the fact that the second derivative of \( f \) at \( x = \frac{18}{5} \) is negative to determine the nature of this critical number, using the Second Derivative Test. (Indeed, \( f''(x) = -2x^{-4/3} - (10/9)x^{-1/3} \), which is negative for all positive \( x \).) However, this test fails to give us any characterization of \( x = 0 \), since \( f'' \) is discontinuous here (a fact we can anticipate, since \( f' \) is undefined at \( 0 \)); thus, we’d still have to use the First Derivative Test in this case.

(b) Find the absolute maximum and minimum values of \( f \) on the interval \([1, 8]\). Justify completely.  

(Hint: it might be useful to note that \( f(x) = x^{2/3}(9 - x) \).)

(3 points) We can apply the Closed Interval Method. Since the critical number \( x = 0 \) is not in this interval, we need only compute the values \( f(1), f(8), \) and \( f(18/5) \); we find

\[
f(1) = 8, \quad f(8) = 8^{2/3}(9 - 8) = 4, \quad f\left(\frac{18}{5}\right) = \left(\frac{18}{5}\right)^{2/3}\left(9 - \frac{18}{5}\right) = \left(\frac{18}{5}\right)^{2/3}\left(\frac{27}{5}\right).
\]

Now although comparing \( f\left(\frac{18}{5}\right) \) against the other two values initially seems daunting, recall that \( f \) is increasing for all \( x \) in \([0, \frac{18}{5}]\), and thus \( f(1) < f\left(\frac{18}{5}\right) \), which guarantees that \( f\left(\frac{18}{5}\right) \) is the largest of the three values above. The smallest is \( f(8) = 4 \), so we conclude that on \([1, 8]\), the absolute maximum value of \( f \) is \( \left(\frac{18}{5}\right)^{2/3}\left(\frac{27}{5}\right) \), and the absolute minimum value of \( f \) is 4.

**Notes:** One could alternatively observe that \( c = \frac{18}{5} \) is the sole critical number in the interval \([1, 8]\), and that we’ve already determined that \( f \) has a local maximum here. Thus, by the “Local to Global” principle, in fact \( f \) has its absolute maximum on \([1, 8]\) here as well.

Incidentally, it’s actually quite simple to directly determine that \( f\left(\frac{18}{5}\right) \) is greater than \( f(1) = 8 \), without a calculator: since \( \frac{18}{5} > 3 \) and \( \frac{27}{5} > 5 \), we have

\[
f\left(\frac{18}{5}\right) = \left(\frac{18}{5}\right)^{2/3}\left(\frac{27}{5}\right) > 3^{2/3} \cdot 5 = 9^{1/3} \cdot 5 > 8^{1/3} \cdot 5 = 2 \cdot 5 = 10 > 8 = f(1)
\]
7. (17 points) Consider the function \( g(x) = xe^{-x} \).

(a) Determine if \( g \) has any asymptotes (horizontal and vertical), with complete reasoning. If there is any vertical asymptote, compute both corresponding one-sided limits.

(4 points) Let us compute the horizontal and vertical asymptotes. Recall that a horizontal asymptote corresponds to the limits \( \lim_{x \to \infty} g(x) \) and \( \lim_{x \to -\infty} g(x) \), if either exists, while the vertical asymptotes correspond to values of \( a \) (if any) such that the limit of \( g(x) \) as \( x \) approaches \( a \) from at least one side is \(+\infty\) or \(-\infty\).

**Horizontal asymptotes:** Let us first calculate the horizontal asymptotes. We have that

\[
\lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{x}{e^x}
\]

This is in the form \( \frac{\infty}{\infty} \), so we can use l’Hôpital’s rule. Thus, we get that

\[
\lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{d}{dx} xe^x = \lim_{x \to \infty} \frac{1}{e^x} = 0
\]

since \( \lim_{x \to \infty} e^x = \infty \). Thus, we get the horizontal asymptote \( y = 0 \) as \( x \to \infty \).

Let us now check for an asymptote at \( -\infty \). First note that

\[
\lim_{x \to -\infty} e^{-x} = +\infty.
\]

Thus,

\[
\lim_{x \to -\infty} xe^{-x} = -\infty,
\]

because the first factor approaches \(-\infty\) and the second approaches \(+\infty\). Therefore, there’s no horizontal asymptote at \( -\infty \).

**Vertical Asymptotes:** Since the function \( g(x) = xe^{-x} \) is defined and continuous for all values of \( x \), it follows that there are no vertical asymptotes.

Thus, the only asymptote is the horizontal asymptote \( y = 0 \) as \( x \to \infty \).

(b) On what interval(s) is \( g \) increasing? decreasing? Explain completely.

(4 points) \( g \) is increasing whenever \( g'(x) > 0 \), and is decreasing whenever \( g'(x) < 0 \). We have that

\[
g'(x) = \frac{d}{dx} (xe^{-x}) = \frac{d}{dx} xe^{-x} + \left( \frac{d}{dx} x \right) e^{-x} = e^{-x} - xe^{-x} = e^{-x}(1 - x)
\]

Thus, \( g'(x) = e^{-x}(1 - x) \). Since \( e^{-x} \) is always positive, we just need to examine \( 1 - x \), which is clearly positive when \( x < 1 \) and negative when \( x > 1 \). Thus, we see that \( g(x) \) is increasing on \((-\infty, 1)\) and decreasing on \((1, \infty)\).
(c) On what interval(s) is $g$ concave up? concave down? Explain completely.

(4 points) $g$ is concave up whenever $g''(x) > 0$, and is concave down whenever $g''(x) < 0$. Since $g'(x) = (1 - x)e^{-x}$, we get that

$$g''(x) = \frac{d}{dx}(1 - x)e^{-x} = (1 - x)(-e^{-x}) - e^{-x} = (x - 2)e^{-x}.$$

Similarly to above, we see that $g''(x)$ is positive whenever $x - 2$ is positive, and is negative whenever $x - 2$ is negative. Thus, we see that $g(x)$ is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

(d) Using the information you’ve found, sketch the graph $y = g(x)$. Label and provide the $(x, y)$ coordinates of any local extrema and inflection points.

(5 points)

![Graph of $y = xe^{-x}$ with labeled points and asymptotes](image_url)
8. (10 points) At any point near a fixed electric charge, the electrostatic potential due to the charge is equal to the amount of charge (in coulombs) divided by the distance to the charge (in centimeters). Suppose two charges, one measuring 80 coulombs and the other 20 coulombs, are placed 10 cm apart. At what point on the line between the charges is the total electrostatic potential minimized? Justify completely.

Let’s call the charge with 80 coulombs A, and the other one B. Given a point on the line between A and B, let \( x \) be the distance from the point to A, and so the distance from the point to B is \( 10 - x \). Then the total electrostatic potential is given by

\[
C(x) = \frac{80}{x} + \frac{20}{10 - x}
\]

The domain of \( C \) is \((0, 10)\), because the point must lie on the line segment between A and B, and \( C(x) \) is undefined when \( x = 0 \) or \( x = 10 \). To find the absolute minimum value of \( C \), we first identify all of its critical numbers:

\[
C'(x) = -\frac{80}{x^2} + \frac{20}{(10 - x)^2} = \frac{-80(10 - x)^2 + 20x^2}{x^2(10 - x)^2}
\]

Setting \( C'(x) = 0 \), we get

\[
-80(10 - x)^2 + 20x^2 = 0
\]

\[
\Rightarrow 3x^2 - 80x + 400 = 0
\]

\[
\Rightarrow (3x - 20)(x - 20) = 0
\]

So \( x = 20/3 \) or 20, but only \( x = 20/3 \) is in the domain of \( C \). Next, we look for points where \( C'(x) \) is undefined. The only places where \( C' \) is undefined is \( x = 0 \) or 10, neither of which is in the domain of \( C \). So we have only one critical number, \( x = 20/3 \).

Finally, we need to verify \( x = 20/3 \) is the point of absolute minimum for \( C \). First we use the Growth Criterion:

\[
\lim_{x \to 0^+} C(x) = \infty
\]

\[
\lim_{x \to 10^-} C(x) = \infty
\]

Hence by continuity, the function \( C \) must have an absolute minimum on its domain, which must occur at a critical number; this must be at \( x = 20/3 \), the sole critical number on the domain of \( C \). Alternatively, we can use the First Derivative Test. When \( x < 20/3 \), \((3x - 20)(x - 20) > 0\), and when \( 20/3 < x < 10 \), \((3x - 20)(x - 20) < 0\). Since \( x^2(10 - x)^2 > 0 \) on the domain of \( C \), we conclude

\[
C'(x) < 0 \text{ when } 0 < x < \frac{20}{3}
\]

\[
C'(x) > 0 \text{ when } \frac{20}{3} < x < 10
\]

As before, we see that \( C \) must achieve its absolute minimum at \( x = \frac{20}{3} \). That is, the position on the line between A and B that is a distance \( \frac{20}{3} \) cm away from A has the smallest electrostatic potential.