Math 41 Final Exam — December 6, 2010

Name: ______________________________________ **SUID#:** _____________

Circle your section:				
Olena Bormashenko 03 (11-11:50am) 06 (1:15-2:05pm)	Ulrik Buchholtz 07 (10-10:50am) 09 (2:15-3:05pm)	John Jiang 02 (1:15-2:05pm) 08 (10-10:50am)	Michael Lipnowski 04 (1:15-2:05pm) 10 (11-11:50am)	Jonathan Lee ACE

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers.
- You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- Please check that your copy of this exam contains 15 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	**Total**
Points:	10	15	12	11	10	11	12	10	15	22	12	140
Score:												

1. (10 points) Find each of the following limits, with justification. If there is an infinite limit, then explain whether it is ∞ or $-\infty$.

   (a) $\lim_{x \to -\infty} x^3 + \sqrt{x^6 + x^3}$

   (b) $\lim_{x \to \infty} x^{x - x^2}$

2. (15 points) In each part below, use the method of your choice to find the derivative. Show the steps in your computations.

   (a) Find $\dfrac{dy}{dx}$ if $y = \dfrac{e^x \cdot 5^{(x^2)} \cdot 11^x}{4^{(x+8)} \cdot 3^{(x^3)}}$

   (b) Find $\dfrac{dy}{dx}$ if $xy = \sin(\cos y)$

(c) Find $g'(x)$ if $g(x) = \int_{x^2}^{\ln x} \arctan t \, dt$

3. (12 points) Suppose that when a gas is placed under certain special conditions, its pressure P (in kPa) and volume V (in liters) obey the equation

$$PV^3 = 270.$$

(a) Pressure is measured with an instrument to be 10 kPa, but the instrument guarantees accuracy only up to ± 0.1 kPa. Use linear approximation to estimate an interval $[a, b]$ in which the value of volume lies; show your reasoning.

For easy reference, the pressure-volume equation is: $PV^3 = 270$.

(b) Now suppose that the pressure is growing at a rate of 2 kPa/min. How fast is the volume of the gas changing when the pressure is exactly 10 kPa?

4. (11 points) Let $f(x) = x^4 - 3x$ and $g(x) = x^2 - x + 1$; suppose we wish to locate points where the curves $y = f(x)$ and $y = g(x)$ intersect.

   (a) Find a single equation satisfied by the x-value for a point of intersection of the graphs of f and g, and explain carefully why this equation has at least two solutions.

   (b) Suppose Newton's method is used to estimate a solution to the equation of part (a) with initial guess $x_1 = -1$. Find the value of x_2.

5. (10 points) An amphibious vehicle requires 5 minutes to travel each mile over land, but only 4 minutes to travel each mile in the water. The vehicle is currently located on water, in a canal that runs in the east-west direction. Its destination point is on land, 3 miles due east and 2 miles due north of its current position. What route will minimize the time required for the vehicle to reach its destination? (You should assume that the time required for the vehicle to transition from water to land is negligible, and that the canal's width is also negligible.) Justify your answer completely.

6. (11 points)

(a) Give a precise statement of the Mean Value Theorem.

(b) Suppose g is a twice differentiable function on $[1, 5]$ satisfying $g(1) = 3$, $g(3) = 7$, and $g(5) = 11$. Show that g'' is zero at some point in the interval $[1, 5]$.

7. (12 points) In the small country of Calasia, the birth and death rates are in balance, which means that the growth rate of the population is the same as its migration rate. (Recall that *migration* is simply the act of people moving in or out of the country.) Let $M(t)$ be the migration rate, in thousands of people per year, at the year t, where t is measured in years since 1980.

   A table of some values for $M(t)$ is given below.

t	0	5	10	15	20	25
$M(t)$	-25	-5	5	-10	-5	15

   (a) Without calculating it, what does the quantity $\int_{20}^{30} M(t)dt$ represent? Express your answer in terms relevant to this situation, and make it understandable to someone who does not know any calculus; be sure to use any units that are appropriate, and also explain what the sign of this quantity would signify.

   (b) Use the *Midpoint Rule* with $n = 2$ to estimate $\int_{5}^{25} M(t)dt$; give your answer as an expression in terms of numbers alone, but you do not have to simplify it.

   (c) Use the *Left Endpoint Rule* with $n = 6$ to estimate $\int_{0}^{30} M(t)dt$; again give your answer in terms of numbers alone.

8. (10 points) Mark each statement below as *true* or *false* by circling **T** or **F**. *No justification is necessary.* All instances of a function f or g refer to a function that is *continuous on its domain.*

T **F** $\int_a^b f(x)\,dx$ measures the total area between $y = f(x)$, $y = 0$, $x = a$, and $x = b$.

T **F** The difference of an antiderivative for f and an antiderivative for g is an antiderivative for $f - g$.

T **F** $\int_0^1 x^2\,dx = \frac{1}{3} + C$ for some unknown constant C.

T **F** If F and G are two antiderivatives of f on $[a, b]$, then $F(x) = G(x) + C$ for some constant C.

T **F** The quotient of an antiderivative for f and an antiderivative for g is an antiderivative for f/g.

T **F** If $f(x)$ is a differentiable function on $[a, b]$, then f has an antiderivative on $[a, b]$.

T **F** If F and G are two antiderivatives of f on $[a, b]$, then the graph of G is a horizontal shift of the graph of F.

T **F** If a is a point in the domain of f, then for every positive ϵ, there is some $\delta > 0$ such that

$$\text{whenever} \quad 0 < |h - 0| < \delta, \quad \text{then} \quad \left| \frac{1}{h} \int_a^{a+h} f(t)\,dt - f(a) \right| < \epsilon.$$

T **F** If $\int_0^1 7f(x)\,dx = 7$, then $\int_0^1 f(x)\,dx = 1$.

T **F** If $\int_0^1 f(x)\,dx = 4$, then $\int_0^1 \sqrt{f(x)}\,dx = 2$.

9. (15 points)

   (a) Suppose $f(x) = x^2$. Let R be the region in the xy-plane bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = 2$, and $x = 3$. Find the area of R by evaluating the limit of a Riemann sum that uses the *Right Endpoint Rule*; show all reasoning.

(b) Express the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{4}{3n}}{1+\frac{4i}{3n}}$$

as a definite integral, and then compute its value using the Evaluation Theorem.

10. (22 points) Show all reasoning when solving each of the problems below.

(a) $\displaystyle\int \left(\frac{1}{\sqrt{1-x^2}} + 2^x + \frac{x^3}{3} - \cos(\pi) \right) dx$

(b) Find a function $F(x)$ such that $F(-1) = 4$ and $F'(x) = x + |x|$ for all x.

(c) $\displaystyle\int \frac{\cos(\tan(x))}{\cos^2(x)}\,dx$

(d) $\displaystyle\int_1^e x\ln(x^2)\,dx$

11. (12 points) Suppose all that is known about the function f is that

$$x - x^2 + 1 \leq f(x) \leq x + x^2 + 1 \quad \text{for all } x.$$

(a) Find $f(0)$.

(b) Determine whether f is differentiable at $x = 0$, using the limit definition of the derivative.

(c) Show that if f is integrable, then $\dfrac{3}{2} \leq \displaystyle\int_{-1}^{2} f(x)\,dx \leq \dfrac{15}{2}$.

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Formulas for Reference

Geometric Formulas:

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3, \qquad SA_{\text{sphere}} = 4\pi r^2, \qquad A_{\text{circle}} = \pi r^2$$

$$V_{\text{cylinder}} = \pi r^2 h, \qquad V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

Summation Formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$