1. (10 points) Find each of the following limits, with justification. If there is an infinite limit, then explain whether it is $\infty$ or $-\infty$.

(a) $\lim_{x \to -\infty} x^3 + \sqrt{x^6 + x^3}$

(5 points) Note that the expression makes sense for $x \leq -1$ (and $x \geq 0$). Note also that for $x \leq -1$ we have $\sqrt{x^6} = -x^3$ (the square root is positive). Now we calculate:

$$\lim_{x \to -\infty} (x^3 + \sqrt{x^6 + x^3}) = \lim_{x \to -\infty} (x^3 + \sqrt{x^6(1 + x^{-3})})$$

$$= \lim_{x \to -\infty} x^3 \left(1 - \sqrt{1 + x^{-3}}\right)$$

$$= \lim_{x \to -\infty} \frac{x^3(1 - \sqrt{1 + x^{-3}})(1 + \sqrt{1 + x^{-3}})}{1 + \sqrt{1 + x^{-3}}}$$

$$= \lim_{x \to -\infty} \frac{x^3(1 - (1 + x^{-3}))}{1 + \sqrt{1 + x^{-3}}}$$

$$= \lim_{x \to -\infty} \frac{-1}{1 + \sqrt{1 + x^{-3}}}$$

$$= -\frac{1}{2}$$

*Alternatives:* Substitute $-x$ for $x$ and look at the limit as $x \to \infty$, then proceed with calculation.

(b) $\lim_{x \to \infty} x^{x-x^2}$

(5 points) To investigate this limit, we take the natural logarithm of the expression, and get

$$\ln \left(x^{x-x^2}\right) = (x-x^2) \ln x = x(1-x) \ln x.$$

The limits of these factors are $\infty$, $-\infty$ and $\infty$, respectively, as $x \to \infty$. By laws of infinite limits, the limit of the product is then $-\infty$. Now we use continuity of the exponential function together with the observation that $\lim_{y \to -\infty} e^y = 0$, to conclude that

$$\lim_{x \to \infty} x^{x-x^2} = 0.$$
2. (15 points) In each part below, use the method of your choice to find the derivative. Show the steps in your computations.

(a) Find \( \frac{dy}{dx} \) if \( y = \frac{e^x \cdot 5^{(x^2)} \cdot 11^x}{4^{(x+8)} \cdot 3^{(x^3)}} \)

(5 points) Taking logarithms of both sides and applying logarithm rules gives

\[
\ln(y) = \ln(e^x) + \ln(5^{x^2}) + \ln(11^x) - \ln(4^{x+8}) - \ln(3^{x^3})
\]

\[
= x + x^2 \ln 5 + x \ln 11 - (x + 8) \ln 4 - x^3 \ln 3
\]

So taking the derivative of both sides gives

\[
\frac{y'}{y} = 1 + 2x \ln 5 + x \ln 11 - \ln 4 - 3x^2 \ln 3
\]

and so

\[
y' = (1 + 2x \ln 5 + x \ln 11 - \ln 4 - 3x^2 \ln 3) \cdot \frac{e^x \cdot 5^{x^2} \cdot 11^x}{4^{x+8} \cdot 3^{x^3}}.
\]

(b) Find \( \frac{dy}{dx} \) if \( xy = \sin(\cos y) \)

(5 points) Taking derivatives of both sides with respect to \( x \), we get

\[
x y' + y = (\cos(\cos y)) \cdot (-\sin y) \cdot y'.
\]

Rearranging (collecting all of the \( y' \) terms together) gives

\[
[x + (\cos(\cos y)) \cdot \sin y] \cdot y' = -y
\]

and solving for \( y' \) gives

\[
y' = \frac{-y}{x + (\cos(\cos y)) \sin y}.
\]
(c) Find $g'(x)$ if $g(x) = \int_{x^2}^{\ln x} \arctan t \, dt$

(5 points) Let $F$ be any antiderivative of $\arctan t$. By the Evaluation Theorem, the integral in the question equals $F(\ln x) - F(x^2)$. Taking the derivative using the chain rule (and remembering that, by definition of antiderivative, $F'(t) = \arctan t$), we get

$$g'(x) = \left(\frac{1}{x}\right) \cdot F'(\ln x) - 2xF'(x^2)$$

$$= \frac{\arctan(\ln x)}{x} - 2x \arctan(x^2).$$
3. (12 points) Suppose that when a gas is placed under certain special conditions, its pressure $P$ (in kPa) and volume $V$ (in liters) obey the equation

$$PV^3 = 270.$$ 

(a) Pressure is measured with an instrument to be 10 kPa, but the instrument guarantees accuracy only up to $\pm 0.1$ kPa. Use linear approximation to estimate an interval $[a, b]$ in which the value of volume lies; show your reasoning.

(6 points) Since we measured the pressure and want to calculate the volume, we solve the equation to find $V$ in terms of $P$,

$$V = \left(\frac{270}{P}\right)^{1/3} = 3\sqrt[3]{10}P^{-1/3}.$$ 

Thus, when $P = 10$, we have $V = 3$. The estimate the error, we differentiate with respect to $P$ to get

$$\frac{dV}{dP} = -\frac{3}{\sqrt[3]{10}}P^{-4/3}.$$ 

Thus, when $P = 10$, we have $dV/dP = -\sqrt[3]{10}/(10\sqrt[3]{10}) = -1/10$. Using linear approximation, we know that $V$ is approximately $3 - 1/10(P - 10)$ for $P$ near 10. Setting $P = 9.9$ and $P = 10.1$, this gives the bounding interval $[2.99, 3.01]$.

Alternatively: Using $dP = 0.1$, the differential of $V$ at $P = 3$ is $dV = (dV/dP) dP = -(1/10) 0.1 = -1/100$, so we estimate that $V = 3 \pm 1/100$, getting the same interval as above.

(b) Now suppose that the pressure is growing at a rate of 2 kPa/min. How fast is the volume of the gas changing when the pressure is exactly 10 kPa?

(6 points) We implicitly differentiate the original equation, $PV^3 = 270$, with respect to time and get

$$\frac{dP}{dt}V^3 + 3PV^2\frac{dV}{dt} = 0.$$ 

Plugging in the numbers $P = 10$, $V = 3$ and $dP/dt = 2$, we get

$$2(3)^3 + 3(10)(3)^2\frac{dV}{dt} = 0.$$ 

Solving, we get $dV/dt = -1/5$, so the volume is shrinking at a rate of 0.2 liters/min at that instant.

Alternatively: Using $dV/dP$ from part (a) with the chain rule, we get

$$\frac{dV}{dt} = \frac{dV}{dP} \frac{dP}{dt}.$$ 

When $P = 10$ we know that $dV/dP = -1/10$, so $dV/dt = -2/10 = -1/5$. 

4. (11 points) Let \( f(x) = x^4 - 3x \) and \( g(x) = x^2 - x + 1 \); suppose we wish to locate points where the curves \( y = f(x) \) and \( y = g(x) \) intersect.

(a) Find a single equation satisfied by the \( x \)-value for a point of intersection of the graphs of \( f \) and \( g \), and explain carefully why this equation has at least two solutions.

(6 points) The graphs intersect precisely when \( f(x) = g(x) \), that is, when \( h(x) = f(x) - g(x) \) has an \( x \)-intercept. The equation \( h(x) = 0 \) is

\[
\begin{align*}
x^4 - x^2 - 2x - 1 &= 0.
\end{align*}
\]

We must now explain why \( h(x) = 0 \) has at least two solutions. We can use the Intermediate Value Theorem since \( h \) is continuous. Note that \( h(0) = -1 \), which is negative. Since \( h(x) \) tends to \( +\infty \) as both \( x \to -\infty \) and \( x \to +\infty \), we conclude by the Intermediate Value Theorem, there is a least one solution with \( x \) negative and one solution with \( x \) positive.

More precisely, we have that \( h(-1) = 1 \) (positive), \( h(0) = -1 \) (negative), \( h(1) = -3 \) (negative) and \( h(2) = 7 \) (positive), so in fact there is a least one solution in each of the intervals \((-1, 0)\) and \((1, 2)\).

**Alternative:** We can solve \( h(x) = 0 \) directly, since

\[
h(x) = 0 \iff x^4 = (x + 1)^2 \iff x^2 = |x + 1| = \begin{cases} x + 1, & \text{for } x \geq -1 \\ -x - 1, & \text{for } x < -1. \end{cases}
\]

The second branch doesn’t have solutions (the discriminant is negative), while the first branch has the solutions \((1 \pm \sqrt{5})/2\). Since these are both \( \geq -1 \), we conclude that \( h(x) = 0 \) has exactly two solutions.

**Third variation:** Another way to solve \( h(x) = 0 \) is to note that \( h \) factors as

\[
h(x) = x^4 - (x + 1)^2 = (x^2 + x + 1)(x^2 - x - 1).
\]

The first factor has no roots, while the second has \((1 \pm \sqrt{5})/2\).

(b) Suppose Newton’s method is used to estimate a solution to the equation of part (a) with initial guess \( x_1 = -1 \). Find the value of \( x_2 \).

(5 points) With \( h(x) = x^4 - x^2 - 2x - 1 \), we get \( h'(x) = 4x^3 - 2x - 2 \). Newton’s method iterates guesses as

\[
x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}.
\]

We have that \( h(-1) = 1 \) and \( h'(-1) = -4 \), so we get

\[
x_2 = -1 - \frac{1}{-4} = \frac{3}{4}.
\]
5. (10 points) An amphibious vehicle requires 5 minutes to travel each mile on land, but only 4 minutes to travel each mile in the water. The vehicle is currently located on water, in a canal that runs in the east-west direction. Its destination point is on land, 3 miles due east and 2 miles due north of its current position. What route will minimize the time required for the vehicle to reach its destination? (You should assume that the time required for the vehicle to transition from water to land is negligible, and that the canal’s width is also negligible.) Justify your answer completely.

As usual, let us first draw a picture representing the situation. We have a canal of negligible width running in an east-west direction, and an amphibious vehicle in the canal. The destination is 3 miles to the east and 2 miles to the north of the vehicle. Since the vehicle is faster in water than on land, the best route will clearly head for some time along the canal, go on land, and then go straight to the destination (straight lines are assumed because they are the fastest way to travel.)

This describes the following diagram:

![Diagram](image)

where the vehicle first travels the road labelled with $3 - x$ and then travels the road labelled with $y$. Let $T(x)$ be the amount of time the vehicle takes if it travels the above route. Since the vehicle takes 4 minutes to travel a mile by water and 5 minutes to travel a mile by land, we get that

$$T(x) = 4(3 - x) + 5y$$

and since the Pythagorean theorem states that

$$y = \sqrt{x^2 + 4}$$

we get that

$$T(x) = 4(3 - x) + 5\sqrt{x^2 + 4}$$

We now need to find $x$ that will minimize the above expression. As usual, we first find $T'(x)$ and use that to find critical points. Rewriting a little, we have that

$$T(x) = 12 - 4x + 5(x^2 + 4)^{1/2}$$

and thus

$$T'(x) = -4 + 5 \cdot \frac{1}{2} (x^2 + 4)^{-1/2} \cdot 2x$$

$$= -4 + \frac{5x}{\sqrt{x^2 + 4}}$$

Recall that critical points are places where $T'(x)$ is either undefined or 0. Since $x^2$ is positive, $\sqrt{x^2 + 4}$ is never 0. Thus, $T'(x)$ is defined everywhere. Let us now solve for all $x$ such that $T'(x) = 0$.

$$0 = T'(x) = -4 + \frac{5x}{\sqrt{x^2 + 4}}$$

$$\Rightarrow 4 = \frac{5x}{\sqrt{x^2 + 4}}$$

$$\Rightarrow 5x = 4\sqrt{x^2 + 4}$$
Now, squaring both sides, we get

\[ 25x^2 = 16(x^2 + 4) \]
\[ \Rightarrow 25x^2 = 16x^2 + 64 \]
\[ \Rightarrow x^2 = \frac{64}{9} \]
\[ \Rightarrow x = \pm \frac{8}{3} \]

It is clear from the diagram that \( x \) has to be between 0 and 3, so the only solution in the domain is \( x = \frac{8}{3} \). We can now either use the first derivative test, the second derivative test, or the closed interval method to check whether \( \frac{8}{3} \) is a minimum or a maximum. Here, we will show the first derivative test, although all of them will work out. Note that there exists only one critical point on \([0, +\infty)\) for the \( T(x) \) defined above, and that

\[ T'(0) = -4 + \frac{5 \cdot 0}{\sqrt{0^2 + 4}} = -4 \]
\[ T'(4) = -4 + \frac{5 \cdot 4}{\sqrt{4^2 + 4}} = -4 + \frac{20}{\sqrt{20}} = -4 + \sqrt{20} > 0 \]

Thus, we see that \( T(x) \) is decreasing on \([0, \frac{8}{3})\) and is increasing on \((\frac{8}{3}, +\infty)\), so \( x = \frac{8}{3} \) is a local minimum. For this \( x \), we can calculate that

\[ y = \sqrt{\frac{64}{9} + 4} = \frac{10}{3} \]

Therefore, the conclusion is that the correct route for the amphibious vehicle is to travel \( \frac{1}{3} \) miles on water and then travel \( \frac{10}{3} \) miles on land.
6. (11 points)

(a) Give a precise statement of the Mean Value Theorem.

(3 points) Suppose \( a < b \), and let \( f \) be a continuous function on \([a, b]\) which is differentiable on \((a, b)\). Then there is some \( c \in [a, b] \) for which
\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

(b) Suppose \( g \) is a twice differentiable function on \([1, 5]\) satisfying \( g(1) = 3 \), \( g(3) = 7 \), and \( g(5) = 11 \). Show that \( g'' \) is zero at some point in the interval \([1, 5]\).

(8 points) \( g \) is a continuous function on the interval \([1, 3]\) which is differentiable on \((1, 3)\). Thus, by the MVT, there is some \( c_1 \in (1, 3) \) such that
\[
g'(c_1) = \frac{g(3) - g(1)}{3 - 1} = \frac{7 - 3}{3 - 1} = 2.
\]

Similarly, there is some \( c_2 \in [3, 5] \) for which
\[
g'(c_2) = \frac{g(5) - g(3)}{5 - 3} = \frac{11 - 7}{3 - 1} = 2.
\]

But \( c_1 < c_2 \). Thus, applying the MVT one more time, we can find some \( d \in (c_1, c_2) \) for which
\[
g''(d) = \frac{g'(c_2) - g'(c_1)}{c_2 - c_1} = \frac{2 - 2}{c_2 - c_1} = 0.
\]
7. (12 points) In the small country of Calasia, the birth and death rates are in balance, which means that the growth rate of the population is the same as its migration rate. (Recall that \textit{migration} is simply the act of people moving in or out of the country.) Let \( M(t) \) be the migration rate, in thousands of people per year, at the year \( t \), where \( t \) is measured in years since 1980.

A table of some values for \( M(t) \) is given below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M(t) )</td>
<td>-25</td>
<td>-5</td>
<td>5</td>
<td>-10</td>
<td>-5</td>
<td>15</td>
</tr>
</tbody>
</table>

(a) Without calculating it, what does the quantity \( \int_{20}^{30} M(t) \, dt \) represent? Express your answer in terms relevant to this situation, and make it understandable to someone who does not know any calculus; be sure to use any units that are appropriate, and also explain what the sign of this quantity would signify.

\( \int_{20}^{30} M(t) \, dt \) is the net change of population, measured in thousands, in Calasia from the year 2000 to the year 2010. You can also call this the net number of people migrating into Calasia between those years. The years are 2000 and 2010 instead of 20 and 30 because \( M(t) \) measures the number of people migrating \( t \) years after 1980.

As noted above, the units here are ‘thousands of people.’ A negative sign would mean that the total population decreased between 2000 and 2010, whereas a positive sign would mean that the total population increased between 2000 and 2010.

(b) Use the \textit{Midpoint Rule} with \( n = 2 \) to estimate \( \int_{5}^{25} M(t) \, dt \); give your answer as an expression in terms of numbers alone, but you do not have to simplify it.

Using the Midpoint Rule with \( n = 2 \), we get that \( \Delta x = \frac{25-5}{2} = 10 \), and thus \( x_0 = 5, x_1 = 15 \) and \( x_2 = 25 \). Thus, the two midpoints are \( \frac{x_0+x_1}{2} = 10 \) and \( \frac{x_1+x_2}{2} = 20 \). Therefore,

\[
\int_{5}^{25} M(t) \, dt \approx 10 \cdot (M(10) + M(20)) = 10 \cdot (5 + (-5)) = 0
\]

(c) Use the \textit{Left Endpoint Rule} with \( n = 6 \) to estimate \( \int_{0}^{30} M(t) \, dt \); again give your answer in terms of numbers alone.

Using the Left Endpoint Rule with \( n = 6 \), we get that \( \Delta x = \frac{30-0}{6} = 5 \), and thus \( x_0 = 0, x_1 = 5, x_2 = 10, x_3 = 15, x_4 = 20 \) and \( x_5 = 25 \). Therefore,

\[
\int_{0}^{30} M(t) \, dt \approx 5 \cdot (M(0) + M(5) + M(10) + M(15) + M(20) + M(25))
\]

\[
= 5(-25 + (-5) + 5 + (-10) + (-5) + 15) = -125
\]
8. (10 points) Mark each statement below as true or false by circling T or F. No justification is necessary. All instances of a function \( f \) or \( g \) refer to a function that is continuous on its domain.

\[ \int_{a}^{b} f(x) \, dx \]

measures the total area between \( y = f(x) \), \( y = 0 \), \( x = a \), and \( x = b \).

\( T \) \hspace{1cm} \( F \) \hspace{1cm} False. The definite integral measures the net signed area between \( y = f(x) \), \( y = 0 \), \( x = a \), and \( x = b \).

\[ \int_{0}^{1} x^2 \, dx = \frac{1}{3} + C \]

for some unknown constant \( C \).

\( T \) \hspace{1cm} \( F \) \hspace{1cm} False. The definite integral equals 1/3. There are no constants of integration in definite integrals.

\( T \) \hspace{1cm} \( F \) \hspace{1cm} If \( F \) and \( G \) are two antiderivatives of \( f \) on \([a, b]\), then \( F(x) = G(x) + C \) for some constant \( C \).

\( T \) \hspace{1cm} \( F \) \hspace{1cm} True. This is a consequence of Theorem 1 of Section 4.8.

\( T \) \hspace{1cm} \( F \) \hspace{1cm} The quotient of an antiderivative for \( f \) and an antiderivative for \( g \) is an antiderivative for \( f/g \).

\( F \) \hspace{1cm} False. If \( F' = f \) and \( G' = g \), then by the quotient rule, \( (F/G)' = \frac{F'G - FG'}{G^2} = \frac{fG - Fg}{G^2} \), which in general will not equal \( f/g \). (For example, let \( F(x) = x + 1 \) and \( G(x) = x \), so that \( f = g = 1 \).)

\( T \) \hspace{1cm} \( F \) \hspace{1cm} If \( f(x) \) is a differentiable function on \([a, b]\), then \( f \) has an antiderivative on \([a, b]\).

\( T \) \hspace{1cm} \( F \) \hspace{1cm} True. If \( f(x) \) is differentiable on \([a, b]\), then it is continuous on \([a, b]\). Thus, by part 1 of the Fundamental Theorem of Calculus, \( f \) has an antiderivative on \([a, b]\).

\( T \) \hspace{1cm} \( F \) \hspace{1cm} If \( F \) and \( G \) are two antiderivatives of \( f \) on \([a, b]\), then the graph of \( G \) is a horizontal shift of the graph of \( F \).

\( T \) \hspace{1cm} \( F \) \hspace{1cm} False. If \( F \) and \( G \) are two antiderivatives of \( f \) on \([a, b]\), then the graph of \( G \) is a vertical shift of the graph of \( F \).
If \( a \) is a point in the domain of \( f \), then for every positive \( \epsilon \), there is some \( \delta > 0 \) such that whenever \( 0 < |h - 0| < \delta \), then \( \left| \frac{1}{h} \int_a^{a+h} f(t) \, dt - f(a) \right| < \epsilon \).

True. The statement above is the delta-epsilon formulation of the limit statement

\[
\lim_{h \to 0} \frac{1}{h} \int_a^{a+h} f(t) \, dt = f(a);
\]

to see that this statement is true, use the fact that \( F(x) = \int_a^x f(t) \, dt \) is an antiderivative of \( f(x) \) that satisfies \( F(a) = 0 \): then

\[
\lim_{h \to 0} \frac{1}{h} \int_a^{a+h} f(t) \, dt = \lim_{h \to 0} \frac{F(a + h)}{h} = \lim_{h \to 0} \frac{F(a + h) - F(a)}{h} \quad \text{(because } F(a) = 0 \text{)}
\]

\[
= F'(a) \quad \text{(by the limit definition of the derivative!)}
\]

\[
= f(a) \quad \text{(since } F \text{ is an antiderivative of } f \text{ by FTC)}
\]

(You could also use l'Hôpital's rule to find the value of the limit.)

If \( \int_0^1 7f(x) \, dx = 7 \), then \( \int_0^1 f(x) \, dx = 1 \).

True. \( \int_a^b Cf(x) \, dx = C \int_a^b f(x) \, dx \) for any constant \( C \).

If \( \int_0^1 f(x) \, dx = 4 \), then \( \int_0^1 \sqrt{f(x)} \, dx = 2 \).

False. For example, if \( f(x) = 8x \), then \( \int_0^1 f(x) \, dx = [4x^2]_x=0^1 = 4 \), but

\[
\int_0^1 \sqrt{8x} \, dx = \left[ \sqrt{8 \frac{x^{3/2}}{3/2}} \right]_0^1 \neq 2.
\]
9. (15 points)

(a) Suppose \( f(x) = x^2 \). Let \( R \) be the region in the \( xy \)-plane bounded by the curve \( y = f(x) \) and the lines \( y = 0, x = 2, \) and \( x = 3 \). Find the area of \( R \) by evaluating the limit of a Riemann sum that uses the Right Endpoint Rule; show all reasoning.

(9 points) Since \( f(x) \) is continuous on the interval \([2, 3]\), it is Riemann integrable. Thus we can choose the following sequence of partitions of \([2, 3]\). For each \( n \), we let

\[
I_i^{(n)} = [x_{i-1}^{(n)}, x_i^{(n)}] := [2 + \frac{i-1}{n}, 2 + \frac{i}{n}]
\]

\( i = 1, 2, \ldots, n \). Notice that each subdivision has length \(|I_i^{(n)}| = \frac{1}{n}\), which goes to 0 as \( n \) goes to infinity. Hence the Riemann sum evaluated over such partitions will converge to the actual Riemann integral, which we take to be the definition of the area of \( R \). The right endpoint of each subinterval \( I_i^{(n)} \) is clearly \( x_i^{(n)} = 2 + \frac{i}{n} \). By definition of Riemann integral, we have

\[
R = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i^{(n)})|I_i^{(n)}|
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (2 + \frac{i}{n})^2 \frac{1}{n}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{4i^2}{n^2} + \frac{i^2}{n^3}
\]

By linearity of limit, we evaluate the limit of the three sums separately. Recall the formula for \( \sum_{i=1}^{n} i \) and \( \sum_{i=1}^{n} i^2 \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{4i^2}{n^2} = \lim_{n \to \infty} \frac{4}{n^2} \sum_{i=1}^{n} i = \lim_{n \to \infty} \frac{4}{n^2} \cdot \frac{n(n+1)}{2} = 2
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} i^2 = \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n \to \infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}
\]

So the area \( R = 4 + 2 + \frac{1}{3} = 19/3 \).
(b) Express the limit
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{4}{3n} \frac{1}{1 + \frac{4i}{3n}}
\]
as a definite integral, and then compute its value using the Evaluation Theorem.

(6 points) We need to cast the sum in the following form
\[
\sum_{i=1}^{n} \frac{4}{3n} \frac{1}{1 + \frac{4i}{3n}} = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})
\]
The expression in the summand suggests taking \(x_i - x_{i-1} = \frac{4}{3n}\), \(x_i = 1 + \frac{4i}{3n}\) and \(f(x) = \frac{1}{x}\), as can be verified by substituting into the right hand side above. Furthermore, since \(x_0 = 1\), and \(x_n = 1 + \frac{4n}{3n} = \frac{7}{3}\), we know the Riemann integral is evaluated on the interval \([1,\frac{7}{3}]\), over which \(f(x)\) is Riemann integrable by continuity. Thus we have
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{4}{3n} \frac{1}{1 + \frac{4i}{3n}} = \int_{1}^{\frac{7}{3}} \frac{1}{x} dx
\]
\[
= \ln |x| \bigg|_{1}^{\frac{7}{3}} = \ln(\frac{7}{3}) - \ln 1 = \ln 7 - \ln 3
\]
where the third to last step uses the Evaluation Theorem.
(a) \( \int \left( \frac{1}{\sqrt{1-x^2}} + 2x + \frac{x^3}{3} - \cos(\pi) \right) \, dx \) 

(5 points) 

\[
\int \frac{1}{\sqrt{1-x^2}} + 2x + \frac{x^3}{3} - \cos(\pi) \, dx = \sin^{-1} x + \frac{2x}{\ln 2} + \frac{x^4}{12} + x + C
\]

(b) Find a function \( F(x) \) such that \( F(-1) = 4 \) and \( F'(x) = x + |x| \) for all \( x \).

(6 points)  

\[
F'(x) = x + |x| = \begin{cases} 
  x + x = 2x & x \geq 0 \\
  x - x = 0 & x < 0 
\end{cases}
\]

Antidifferentiating gives 

\[
F(x) = \begin{cases} 
  x^2 + c_1 & x \geq 0 \\
  c_2 & x < 0 
\end{cases}
\]

for some constants \( c_1 \) and \( c_2 \). We must have \( F(-1) = c_2 = 4 \). Also, in order for \( F \) to be differentiable everywhere, it must in particular be continuous at zero, so we have \( c_1 = \lim_{x \to 0^-} F(x) = \lim_{x \to 0^+} F(x) = c_2 = 4 \). This gives 

\[
F(x) = \begin{cases} 
  x^2 + 4 & x \geq 0 \\
  4 & x < 0 
\end{cases}
\]

and it is easy to check that \( F(x) \) is differentiable everywhere.
(c) \[ \int \frac{\cos(\tan(x))}{\cos^2(x)} \, dx \]

(5 points) Let \( u = \tan(x) \); then \( du = \sec^2(x) \, dx = \frac{1}{\cos^2(x)} \, dx \) and

\[
\int \frac{\cos(\tan(x))}{\cos^2(x)} \, dx = \int \cos(u) \, du = \sin(u) + C = \sin(\tan(x)) + C
\]

(d) \[ \int_1^e x \ln(x^2) \, dx \]

(6 points) Integration by parts: let \( u = \ln(x^2) \), \( du = \frac{2x}{x^2} \, dx = \frac{2}{x} \, dx \), \( dv = x \, dx \), \( v = x^2/2 \). Then we have

\[
\int_1^e x \ln(x^2) \, dx = \left[ \ln(x^2) \frac{x^2}{2} - \int \frac{x^2}{2} \frac{2}{x} \, dx \right]_1^e
\]

\[
= \left[ \ln(x^2) \frac{x^2}{2} - \frac{x^2}{2} \right]_1^e
\]

\[
= \ln(e^2) \left( \frac{e^2}{2} \right) - \frac{e^2}{2} - \ln(1) (1/2) + \frac{1}{2}
\]

\[
= \frac{2(e^2)}{2} - \frac{e^2}{2} + \frac{1}{2}
\]

\[
= \frac{e^2}{2} + \frac{1}{2}
\]
11. (12 points) Suppose all that is known about the function \( f \) is that
\[
x - x^2 + 1 \leq f(x) \leq x + x^2 + 1 \quad \text{for all } x.
\]

Let \( g(x) = x - x^2 + 1 \) and \( h(x) = x + x^2 + 1 \); we are given that \( f(x) \) is a function satisfying \( g(x) \leq f(x) \leq h(x) \) for all \( x \).

(a) Find \( f(0) \).

(1 point) By directly substituting in \( x = 0 \), we see that
\[
g(0) = 0 - 0^2 + 1 = 1 \quad \text{and} \quad h(0) = 0 + 0^2 + 1 = 1.
\]

Since \( g(0) \leq f(0) \leq h(0) \), this means that \( f(0) = 1 \).

Note that the squeeze theorem would inform you about \( \lim_{x \to 0} f(x) \), which does not apply here, unless you can assume that the function \( f(x) \) is continuous at \( x = 0 \). (However, continuity is implied by the next part of the question.)

(b) Determine whether \( f \) is differentiable at \( x = 0 \), using the limit definition of the derivative.

(5 points) By the definition of a derivative, we have
\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 1}{x}.
\]

The information provided about \( f(x) \) is sufficient to compute this limit. (Note that l'Hôpital’s rule does not apply here, and will lead to circular arguments that get you nowhere.)

Indeed, subtracting 1 from each side of the relation \( g(x) \leq f(x) \leq h(x) \), we obtain
\[
x - x^2 \leq f(x) - 1 \leq x + x^2.
\]

For \( x \neq 0 \), we may divide through by \( |x| \) to obtain
\[
x/|x| - |x| \leq \frac{f(x) - 1}{|x|} \leq x/|x| + |x|.
\]

Independently checking the cases \( x > 0 \) and \( x < 0 \) shows that this is equivalent to:
\[
1 - |x| \leq \frac{f(x) - 1}{x} \leq 1 + |x|.
\]

By direct substitution of \( x = 0 \), we obtain the limits for the left and right-hand sides:
\[
\lim_{x \to 0} 1 - |x| = 1 \quad \text{and} \quad \lim_{x \to 0} 1 + |x| = 1.
\]

Thus, by the squeeze theorem, the middle limit also exists:
\[
\lim_{x \to 0} \frac{f(x) - 1}{x} = 1.
\]

By the definition of the derivative stated above, this means that \( f'(0) = 1 \); in particular, \( f \) is differentiable at \( x = 0 \).
Further notes: Alternatively, one can argue along the lines that if \( g(x) \leq f(x) \leq h(x) \) for all \( x \), \( g(0) = h(0) \), and \( g'(0) = h'(0) \), then \( f'(0) \) also exists and is equal to \( g'(0) = h'(0) \). This is a true fact not proven in class, so in order to get more than a little partial credit, you need to prove it as follows (and not just casually quote the squeeze theorem):

Let \( L \) denote the common value \( g'(0) = h'(0) \) and let \( y \) denote the common value \( g(0) = f(0) = h(0) \). For any \( \varepsilon > 0 \), we need to show that there is a \( \delta > 0 \) such that

\[
0 < |x| < \delta \quad \text{implies} \quad \left| L - \frac{f(x) - y}{x} \right| < \varepsilon.
\]

By definition of \( g'(0) \), there is a \( \delta_g > 0 \) such that

\[
0 < |x| < \delta_g \quad \text{implies} \quad \left| L - \frac{g(x) - y}{x} \right| < \varepsilon.
\]

Similarly, definition of \( h'(0) \), there is a \( \delta_h > 0 \) such that

\[
0 < |x| < \delta_h \quad \text{implies} \quad \left| L - \frac{h(x) - y}{x} \right| < \varepsilon.
\]

Set \( \delta \) equal to the smaller of \( \delta_g \) and \( \delta_h \). Then, whenever \( 0 < |x| < \delta \), we have

\[
\left| L - \frac{g(x) - y}{x} \right| < \varepsilon \quad \text{and} \quad \left| L - \frac{h(x) - y}{x} \right| < \varepsilon.
\]

Since \( g(x) \leq f(x) \leq h(x) \), the (signed) value \( L - \frac{g(x) - y}{x} \) is bounded on both sides by \( L - \frac{g(x) - y}{x} \) and \( L - \frac{h(x) - y}{x} \) (in some order depending on the sign of \( x \)). Hence,

\[
\left| L - \frac{f(x) - y}{x} \right| \leq \max \left( \left| L - \frac{g(x) - y}{x} \right|, \left| L - \frac{h(x) - y}{x} \right| \right)
\]

\[
= \max(\varepsilon, \varepsilon) = \varepsilon,
\]

which is what we set out to show.

Even more notes: The above argument only works when \( g(0) = h(0) \) and \( g'(0) = h'(0) \) — remove either condition and the argument not only fails, but the “fact” is no longer true.

(c) Show that if \( f \) is integrable, then \( \frac{3}{2} \leq \int_{-1}^{2} f(x) \, dx \leq \frac{15}{2} \).

(6 points) Integrating each term of the relation \( g(x) \leq f(x) \leq h(x) \), we obtain

\[
\int_{-1}^{2} g(x) \, dx \leq \int_{-1}^{2} f(x) \, dx \leq \int_{-1}^{2} h(x) \, dx.
\]

(Note that the comparison property only holds for definite integrals. It does not make sense to compare indefinite integrals, in part because there is no canonical choice of anti-derivative.)

To compute the left-hand side, we see that one anti-derivative of \( g(x) \) is given by:

\[
G(x) = \frac{x^2}{2} - \frac{x^3}{3} + x,
\]
and so

\[
\int_{-1}^{2} g(x) \, dx = G(2) - G(-1) \\
= \left( \frac{2^2}{2} - \frac{2^3}{3} + 2 \right) - \left( \frac{(-1)^2}{2} - \frac{(-1)^3}{3} + (-1) \right) \\
= (2 - 8/3 + 2) - (1/2 + 1/3 - 1) \\
= \frac{3}{2}.
\]

Similarly, to compute the left-hand side, we see that one anti-derivative of \( h(x) \) is given by:

\[
H(x) = \frac{x^2}{2} + \frac{x^3}{3} + x,
\]

and so

\[
\int_{-1}^{2} h(x) \, dx = H(2) - H(-1) \\
= \left( \frac{2^2}{2} + \frac{2^3}{3} + 2 \right) - \left( \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + (-1) \right) \\
= (2 + 8/3 + 2) - (1/2 - 1/3 - 1) \\
= \frac{15}{2}.
\]

Putting these together shows that

\[
\frac{3}{2} \leq \int_{-1}^{2} f(x) \, dx \leq \frac{15}{2}.
\]