1. (15 points) Find each of the following.

(a) \( \int \left( 4t^4 - t^{-1} + \frac{2}{t^2 + 1} \right) dt \)

\[
= \int 4t^4 dt + \int -t^{-1} dt + \int \frac{2}{t^2 + 1} dt \\
= 4\int t^4 dt - \int \frac{1}{t} dt + 2\int \frac{1}{t^2 + 1} dt \\
= \frac{4}{5}t^5 - \ln |t| + 2\tan^{-1} t + C
\]

(b) \( \int \left( \sec x \tan x - \sin x + \frac{x^2 + 1}{3\sqrt{x}} \right) dx \)

\[
= \int \sec x \tan x dx + \int -\sin x dx + \int \left( \frac{x^2}{3\sqrt{x}} + \frac{1}{3\sqrt{x}} \right) dx \\
= \int \sec x \tan x dx + \int -\sin x dx + \frac{1}{3} \int x^{3/2} dx + \frac{1}{3} \int x^{-1/2} dx \\
= \sec x + \cos x + \frac{1}{3} \cdot \frac{2}{5} \cdot x^{5/2} + \frac{1}{3} \cdot 2x^{1/2} + C
\]
1. (15 points) Evaluate the following definite integrals.

(a) \[ \int_0^2 x^2 e^{x^3} \, dx \]
Let \( u = x^3 \)
\[ du = 3x^2 \, dx \quad \Rightarrow \quad \frac{1}{3} du = x^2 \, dx. \]
Also, \( x = 0 \Rightarrow u = 0 \), \( x = 2 \Rightarrow u = 8 \).

\[ \int_0^2 x^2 e^{x^3} \, dx = \int_0^8 e^u \cdot \frac{1}{3} \, du \]
\[ = \frac{1}{3} \int_0^8 e^u \, du = \frac{1}{3} e^u \bigg|_{u=0}^{u=8} = \frac{1}{3}(e^8 - 1) \]

(b) \[ \int_0^1 x^2 e^{3x} \, dx \]

**Int.-by-Parts:** Let \( u = x^2 \)
\[ du = 2x \, dx \]
\[ dv = e^{3x} \, dx \quad v = \frac{1}{3} e^{3x}. \]

\[ \int_0^1 x^2 e^{3x} \, dx = \int_0^1 x^2 \cdot \frac{1}{3} e^{3x} \, dx \]
\[ = \frac{1}{3} \int_0^1 x^2 e^{3x} \, dx - \frac{2}{3} \int_0^1 xe^{3x} \, dx \]

**New int.-by-parts:** \( f = x \)
\[ df = dx \]
\[ dg = e^{3x} \, dx \quad g = \frac{1}{3} e^{3x}. \]

\[ \int_0^1 xe^{3x} \, dx = \frac{1}{3} xe^{3x} \bigg|_0^1 - \frac{2}{3} \left( \int_0^1 x e^{3x} \, dx \right) \]
\[ = \left( \frac{1}{3} e^{3x} \right) \bigg|_0^1 - \frac{2}{3} \left( \int_0^1 x e^{3x} \, dx \right) \]
\[ = \left( \frac{1}{3} e^{3} - \frac{2}{3} e^{3} \right) \bigg|_0^1 = \left( \frac{1}{3} e^{3} - \frac{2}{3} e^{3} \right) = \frac{1}{3} e^{3} - \frac{2}{3} e^{3} \]
2. (15 points)

(a) Let \( f(x) = e^x \). On the graph of \( f \) pictured below, draw the approximating rectangles that are used to estimate the area under the curve between \( x = 0 \) and \( x = 1 \) according to the the Left Endpoint Rule; use \( n = 4 \) rectangles.

(b) Write an expression involving only numbers (including \( e \)) that represents the area estimate using these rectangles. Is your quantity an underestimate or an overestimate of the actual area?

\[
\text{Area} = \text{area}_1 + \text{area}_2 + \text{area}_3 + \text{area}_4 \\
= \text{height}_1 \cdot \text{width} + \text{height}_2 \cdot \text{width} + \text{height}_3 \cdot \text{width} + \text{height}_4 \cdot \text{width} \\
= f(0) \cdot \frac{1}{4} + f\left(\frac{1}{4}\right) \cdot \frac{1}{4} + f\left(\frac{2}{4}\right) \cdot \frac{1}{4} + f\left(\frac{3}{4}\right) \cdot \frac{1}{4} \\
= e^0 \cdot \frac{1}{4} + e^{\frac{1}{4}} \cdot \frac{1}{4} + e^{\frac{2}{4}} \cdot \frac{1}{4} + e^{\frac{3}{4}} \cdot \frac{1}{4} \\
= \left(1 + e^{\frac{1}{4}} + e^{\frac{2}{4}} + e^{\frac{3}{4}}\right) \cdot \frac{1}{4}
\]

This is an \underline{underestimate} of the actual area, since the rectangles lie underneath the curve. (This is due to the fact that \( f(x) = e^x \) is an increasing function.)
(c) Write a mathematical statement that expresses the area under the curve from $x = 0$ to $x = 1$ as a limit, again using the Left Endpoint Rule. Explain any notation you use. (You should not evaluate this limit.)

Approximate area using $n$ rectangles (see sketch)

is $f(0) \cdot \frac{1}{n} + f(\frac{1}{n}) \cdot \frac{1}{n} + \cdots + f(\frac{n-1}{n}) \cdot \frac{1}{n}$, so

the limit expression is:

$$\text{Area} = \lim_{n \to \infty} \left(f(0) \cdot \frac{1}{n} + f(\frac{1}{n}) \cdot \frac{1}{n} + \cdots + f(\frac{n-1}{n}) \cdot \frac{1}{n}\right)$$

$$= \lim_{n \to \infty} \left(e^0 + e^{\frac{1}{n}} + \cdots + e^{\frac{n-1}{n}}\right) \cdot \frac{1}{n} = \lim_{n \to \infty} \left(\sum_{i=0}^{n-1} e^{i/n} \cdot \frac{1}{n}\right)$$

(Any of these expressions is accepted.)

(d) Calculate the value of the definite integral $\int_0^1 e^x \, dx$, simplifying your answer as much as you can.

$$\int_0^1 e^x \, dx = \left[e^x \right]_0^1 = e^1 - e^0 = e - 1$$
3. (30 points) Evaluate the following integrals, showing all of your work.

(a) \( \int \frac{x^3}{\sqrt{1-x^2}} \, dx \)

Let \( u = 1-x^2 \), so \( du = -2x \, dx \).

Then \( \int \frac{x^3}{\sqrt{1-x^2}} \, dx = \int \frac{x^2 \cdot x}{\sqrt{1-x^2}} \, dx = \int \frac{(1-u) \cdot \frac{du}{\sqrt{u}}}{\sqrt{u}} \)

\[ = -\frac{1}{2} \int \frac{1-u}{\sqrt{u}} \, du \]
\[ = -\frac{1}{2} \int \left( \frac{1}{\sqrt{u}} - \frac{u}{\sqrt{u}} \right) \, du \]
\[ = -\frac{1}{2} \int \left( u^{-\frac{1}{2}} - u^\frac{3}{2} \right) \, du \]
\[ = -u^{\frac{1}{2}} + \frac{1}{3} u^{\frac{3}{2}} + C = \sqrt{1-x^2} + \frac{1}{3} (1-x^2)^{\frac{3}{2}} + C \]

Note: there are other equivalent answers that arise using other solution methods (like integration by parts).

(c) \( \int x^2 (\ln x)^3 \, dx \)

Int-by-parts:
\( u = (\ln x)^3 \), \( du = 3(\ln x)^2 \cdot \frac{1}{x} \, dx \)
\( dv = x^2 \, dx \), \( v = \frac{x^3}{3} \)

So, \( \int x^2 (\ln x)^3 \, dx = \frac{x^3}{3} (\ln x)^3 - \int \frac{2}{3} x^2 \ln x \, dx = \frac{x^3}{3} (\ln x)^3 - \frac{2}{3} \int x^2 \ln x \, dx \).

Second int-by-parts:
\( u_2 = \ln x \), \( du_2 = \frac{1}{x} \, dx \)
\( dv_2 = x^2 \, dx \), \( v_2 = \frac{x^3}{3} \)

So, \( \frac{x^3}{3} (\ln x)^3 - \frac{2}{3} \int x^2 \ln x \, dx = \frac{x^3}{3} (\ln x)^3 - \frac{2}{3} \left[ \frac{x^3}{3} \ln x - \frac{x^2}{3} \right] \)
\[ = \frac{x^3}{3} (\ln x)^3 - \frac{2}{9} x^2 \ln x + \frac{2}{27} x^3 + C \]

(a) \( \int x \sin x \, dx \)
4. (16 points) A particle moves along a line, with acceleration (in meters/sec$^2$) as a function of time $t$ (in seconds) given by

\[ a(t) = 2t - 7. \]

Furthermore, at time $t = 1$ second, the particle's velocity is 4 meters per second.

(a) Find the particle's velocity function $v(t)$.

$v(t)$ is antiderivative of $a(t)$:

\[ v(t) = \int a(t) \, dt = \int (2t-7) \, dt = t^2 - 7t + C. \]

Since $v(1) = 4$, we have

\[ 4 = v(1) = 1^2 - 7\cdot1 + C \implies C = 10. \]

So \[ v(t) = t^2 - 7t + 10. \]

(b) What is the particle's net change in position (i.e., its displacement) between the times $t = 1$ and $t = 7$?

By the "net change" theorem, displacement betw. $t=1$ and $t=7$ equals:

Net change in position $s(t)$ between $t=1$ & $t=7$

\[
= \int_1^7 s'(t) \, dt \\
= \int_1^7 v(t) \, dt \\
= \int_1^7 (t^2 - 7t + 10) \, dt \\
= \left[ \frac{t^3}{3} - \frac{7t^2}{2} + 10t \right]_1^7 \\
= \frac{7^3}{3} - \frac{7\cdot7^2}{2} + 10\cdot7 - \left( \frac{1}{3} - \frac{7}{2} + 10 \right) \\
= \boxed{6} \text{ meters (simplification not necessary)}.\]
(c) Find the total distance traveled by the particle between \( t = 1 \) and \( t = 7 \).

We must first determine the intervals when the particle is moving left & right.

When is \( v(t) = 0 \)?
\[
\begin{align*}
  t^2 - 7t + 10 &= 0 \\
  (t-5)(t-2) &= 0 \implies t = 2, 5.
\end{align*}
\]

If \( t < 2 \), then both \( t - 2 \) and \( t - 5 \) are negative, so \( v(t) = \Theta \cdot \Theta \)
\[= \Theta.\]

If \( 2 < t < 5 \), then \( t - 2 \) is positive and \( t - 5 \) is negative, so \( v(t) = \Theta \cdot \Theta \)
\[= \Theta.\]

If \( t > 5 \), then both \( t - 2 \) and \( t - 5 \) are positive, so \( v(t) = \Theta \cdot \Theta = \Theta.\)

Thus, the particle is moving:

- to the right [resulting in a positive displacement] betw. \( t = 1 \) \& \( t = 2 \),
- to the left [resulting in a negative displacement] betw. \( t = 2 \) \& \( t = 5 \), and
- to the right [resulting in a positive displacement] betw. \( t = 5 \) \& \( t = 7 \).

Thus the total distance traveled will be:
\[
\sum_{t=1}^{2} v(t) \, dt + \left| \sum_{t=2}^{5} v(t) \, dt \right| + \left| \sum_{t=5}^{7} v(t) \, dt \right| = \left( \left( \frac{t^3}{3} - \frac{7t^2}{2} + 10t \right) \right)_{1}^{2} + \left| \left( \frac{t^3}{3} - \frac{7t^2}{2} + 10t \right) \right|_{2}^{5} + \left( \left( \frac{t^3}{3} - \frac{7t^2}{2} + 10t \right) \right)_{5}^{7}
\]

Can leave in this form for full credit (not asked to simplify!)
\[
\left\{ \begin{align*}
  \left( \frac{2^3}{3} - \frac{7\cdot2^2}{2} + 10\cdot2 \right) - \left( \frac{1^3}{3} - \frac{7\cdot1^2}{2} + 10\cdot1 \right) + \left| \left( \frac{2^3}{3} - \frac{7\cdot2^2}{2} + 10\cdot2 \right) - \left( \frac{5^3}{3} - \frac{7\cdot5^2}{2} + 10\cdot5 \right) \right|
\end{align*} \right.
\]

\[
+ \left( \frac{7^3}{3} - \frac{7\cdot7^2}{2} + 10\cdot7 \right) - \left( \frac{5^3}{3} - \frac{7\cdot5^2}{2} + 10\cdot5 \right)
\]

\[= \frac{11}{6} + \left| - \frac{9}{2} \right| + \frac{26}{3} = 15 \text{ meters} \]
6. (15 points) For this problem, we define \( g \) to be the function

\[
    g(x) = \int_{-3}^{x} f(t) \, dt, \quad \text{where} \quad f(x) = \begin{cases} 
    x + 1 & -3 \leq x \leq -1 \\
    -\sqrt{1 - x^2} & -1 \leq x \leq 1 \\
    2x - 2 & 1 \leq x \leq 3 \\
    -2x + 10 & 3 \leq x \leq 5
\end{cases}
\]

Below is the graph of the function \( f \):

(a) Find \( g'(-2) \).

\[
    g'(x) = \frac{d}{dx} g(x) = \frac{d}{dx} \int_{-3}^{x} f(t) \, dt = f(x) \quad \text{by the Fundamental Theorem of Calculus, so}
\]

\[
    g'(-2) = f(-2) = \boxed{1}.
\]

(b) On what intervals is the graph of \( g \) concave down?

\[
    g \text{ concave down} \iff g' \text{ is decreasing} \iff f \text{ is decreasing}
\]

So we can use the graph (or the formula) for \( f \) to conclude that the intervals are

\[
    \{ -1 < x < 0 \} \quad \text{and} \quad \{ 3 < x \leq 5 \}.
\]

(i.e., \((-1, 0) \cup (3, 5)\).
For easy reference, here again are the graph of $f$ and the definitions of $f$ and $g$:

$$f(x) = \begin{cases} 
  x + 1 & -3 \leq x \leq -1 \\
  -\sqrt{1 - x^2} & -1 \leq x \leq 1 \\
  2x - 2 & 1 \leq x \leq 3 \\
  -2x + 10 & 3 \leq x \leq 5 
\end{cases}$$

$$g(x) = \int_{-3}^{x} f(t) \, dt$$

(c) Determine the absolute maximum and minimum values of $g$ on the interval $[-3, 5]$.

Critical numbers of $g$: ask when is $g'(x) = 0$ or $g'(x)$ undefined?

Since $g'(x) = f(x)$, the answer is: $f(x) = 0$ when $x = -1, 1, 5$ (and $f$ never undefined).

List of candidates: The x-values we must check are $x = -3, -1, 1, 5$ (because $x = -3$ is an endpoint).

* Test $g(-3) = \int_{-3}^{-1} f(t) \, dt = 0$.
* Test $g(-1) = \int_{-3}^{-1} f(t) \, dt = -(\text{area of triangle (1) above}) = -2$.
* Test $g(1) = \int_{-3}^{1} f(t) \, dt = -(\text{area of triangle (1) + area of semicircle (2)}) = \boxed{-2 - \frac{\pi}{2}}$.
* Test $g(5) = \int_{-3}^{5} f(t) \, dt = -(\text{triangle (1) + semicircle (2)} + \text{triangles (3) & (4)}) = -2 - \frac{\pi}{2} + 8 = \boxed{6 - \frac{\pi}{2}}$.

(d) Find a formula that gives the value of $g(x)$ for any $3 \leq x \leq 5$. (Notice that this is just a small portion of the domain graphed above.)

Several methods can be used. One idea is to write

$$g(x) = \int_{-3}^{x} f(t) \, dt = \int_{-3}^{3} f(t) \, dt + \int_{3}^{x} f(t) \, dt$$

$$= - (\text{area of (1) + (2)}) + (\text{area of (3)}) + \int_{3}^{x} (-2t + 10) \, dt$$

$$= -2 - \frac{\pi}{2} + 4 + \left[ -t^2 + 10t \right]_{t=3}^{t=x} = \boxed{x^2 + 10x - \frac{\pi}{2} - 19}.$$ 

Another is to notice that since $g'(x) = f(x) = -2x + 10$ on the interval $[3, 5]$, we already know that $g(x) = -x^2 + 10x + C$ for some constant $C$, and then use the value $g(5) = 6 - \frac{\pi}{2}$ that was calculated in part (c) to get $C = -\frac{\pi}{2} - 19$. 

6. (14 points) The growth rate, measured in dollars per year, of a certain prestigious university’s tuition can be modeled by the function \( h(t) \), where \( t \) is the number of years since 1990:

\[
h(t) = 1000 \cdot (1.05)^t
\]

(a) Calculate \( \int_{14}^{18} h(t) \, dt \). (You do not need to express your answer in simplest form.)

\[
\int_{14}^{18} h(t) \, dt = \int_{14}^{18} 1000 \cdot (1.05)^t \, dt = 1000 \cdot \left[ \frac{1}{\ln(1.05)} \cdot (1.05)^t \right]_{t=14}^{t=18}
\]

\[
= 1000 \cdot \left( \frac{1}{\ln(1.05)} \cdot (1.05)^{18} - (1.05)^{14} \right)
\]

(b) What are the units of your answer to part (a)? Write a sentence expressing the meaning of the quantity you found.

- \( t \) measured in years, \( h(t) \) measured in dollars/yr, so \( \int_{14}^{18} h(t) \, dt \) is measured in dollars.

- The quantity in part (a) is the total (net) change in tuition between time \( t=14 \) and \( t=18 \); i.e. between 2004 and 2008.

(c) Find \( \frac{d}{dx} \int_x^{15} 1000(1.05)^t \, dt \), showing all steps in your calculation.

\[
\frac{d}{dx} \int_x^{15} 1000(1.05)^t \, dt = \frac{d}{dx} \left( -\int_{15}^x 1000(1.05)^t \, dt \right)
\]

\[
= -\frac{d}{dx} \int_{15}^x 1000(1.05)^t \, dt = -1000(1.05)^x
\]

(By Fund. Thm. of Calc.)
7. (15 points)

(a) Determine \( \frac{d}{dx} \int_{x}^{\arctan x} 2^t \, dt \).

Write \( F(t) \) for an antiderivative of \( 2^t \), so that \( F'(t) = 2^t \).

Then \( \int_{x}^{\arctan x} 2^t \, dt = F(\arctan x) - F(x) \),

so that \( \frac{d}{dx} \int_{x}^{\arctan x} 2^t \, dt = \frac{d}{dx} \left[ F(\arctan x) - F(x) \right] = F'(\arctan x) \cdot \frac{d}{dx}(\arctan x) - F'(x) \)

\[ = \arctan x \cdot \frac{1}{1+x^2} - 2^x. \]

Notice that we never needed to know that \( F(t) = \frac{2^t}{\ln 2} \)!

(b) Find a function \( f \) and a value of the constant \( a \) such that \( \int_{a}^{\infty} f(t) \, dt = 4\sqrt{9+a^2} - 16 \).

Take \( \frac{d}{dx} \) of both sides of the above equation:

\[ \frac{d}{dx} \int_{a}^{x} f(t) \, dt = \frac{d}{dx} \left[ 4\sqrt{9+t^2} - 16 \right] \]

by FTC (Fundamental Theorem of Calculus),

\[ f(x) = 4 \cdot \frac{1}{2} \cdot (9+x^2)^{1/2} \cdot 2x = \frac{4x}{\sqrt{9+x^2}}. \]

and so this gives us function \( f \).

Now take the original equation and set \( x=a \):

\[ \int_{a}^{\infty} f(t) \, dt = 4\sqrt{9+a^2} - 16 \]

\[ \bigg|_{0}^{\infty} = 4\sqrt{9+a^2} - 16 \], and this can now be solved for \( a \),

where we get \( a = \sqrt{7} \) or \(-\sqrt{7} \).
7. (12 points) Let \( g \) be a function for which the following things are true:

- \( g(0) = 2 \),
- the graph of \( g' \), the derivative of \( g \), is shown below:

![Graph of g'](

(a) Apply the Evaluation Theorem to write a simple mathematical statement involving the definite integral \( \int_a^b g'(t) \, dt \).

\[ a \int_a^b g'(t) \, dt = g(b) - g(a) \]

(b) Use \( a = 0 \) and \( b = x \) in your answer to part (a) to obtain an expression for \( g(x) \) in terms of an integral.

\[ \int_0^x g'(t) \, dt = g(x) - g(0) \quad \text{or} \quad g(x) = g(0) + \int_0^x g'(t) \, dt \]

(c) Use part (b) and the graph above to fill in the table of values below for \( g(x) \). In the space below the table, give a sentence (or more if necessary) explaining your reasoning. (You may also use the next page if needed.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>2</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( -\frac{1}{2} )</td>
<td>-1</td>
<td>( -\frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

Regard part (b)'s answer as a formula for \( g(x) \) in terms of \( g(0) \) (which is 2) and the area under the curve \( g' \) between 0 and \( x \).

For example, \( g(1) = g(0) + \int_0^1 g'(t) \, dt = 2 + \left( \text{area under } g' \text{ between 0 and 1} \right) = 2 + -\frac{1}{2} = \frac{3}{2} \), and \( g(2) = g(0) + \int_0^2 g'(t) \, dt = 2 + \left( \text{area under } g' \text{ between 0 and 2} \right) = 2 - \frac{3}{2} = \frac{1}{2} \), etc.
Math 20 Winter 2006 - Exam #2 Solutions

1. (15 points) Find each of the following.

(a) \( \int \left( 3 \csc^2 t + 2^t - \frac{1}{t^2} \right) dt \)

\[
= 3 \int \csc^2 t dt + \int 2^t dt - \int \frac{1}{t^2} dt \\
= -3 \cot t + \frac{2^t}{\ln 2} - \ln |t| + C
\]

(b) \( \int \left( 5(1 - x^2)^{-\frac{1}{2}} + (\sqrt{x})(x^2 - 2x) - 3e^x \right) dx \)

\[
= 5 \int \frac{1}{\sqrt{1 - x^2}} dx + \int (x^{3/2} - 2x^{3/2}) dx - \int 3e^x dx \\
= 5 \int \frac{1}{\sqrt{1 - x^2}} dx + \int x^{3/2} dx - 2\int x^{3/2} dx - 3e^x dx \\
= 5 \sin^{-1} x + \frac{3}{7} x^{7/2} - \frac{4}{5} x^{5/2} - 3e^x + C
\]
2. (15 points)

(a) Let \( f(x) = \sqrt{1-x^2} \). On the axes pictured below, draw the approximating rectangles that are used to estimate the area under the curve between \( x = 0 \) and \( x = 1 \) according to the Right Endpoint Rule; use \( n = 6 \) rectangles.

(b) Write an expression involving only numbers that represents the area estimate using these rectangles. (You do not have to simplify the expression.) Is your quantity an underestimate or an overestimate of the actual area?

\[
\text{Area} = \text{Area}_1 + \text{Area}_2 + \cdots + \text{Area}_6 \\
= \text{width} \cdot \text{height}_1 + \text{width} \cdot \text{height}_2 + \cdots + \text{width} \cdot \text{height}_6 \\
= \text{width} \cdot (\text{height}_1 + \text{height}_2 + \cdots + \text{height}_6) \\
= \frac{1}{6} \cdot \left( f\left(\frac{1}{6}\right) + f\left(\frac{2}{6}\right) + f\left(\frac{3}{6}\right) + f\left(\frac{4}{6}\right) + f\left(\frac{5}{6}\right) + f\left(\frac{6}{6}\right) \right) \\
= \frac{1}{6} \cdot \left( \sqrt{1-\left(\frac{1}{6}\right)^2} + \sqrt{1-\left(\frac{2}{6}\right)^2} + \sqrt{1-\left(\frac{3}{6}\right)^2} + \sqrt{1-\left(\frac{4}{6}\right)^2} + \sqrt{1-\left(\frac{5}{6}\right)^2} + \sqrt{1-\left(\frac{6}{6}\right)^2} \right)
\]

The estimate is an **underestimate** of the actual area, because the rectangles lie underneath the curve (a phenomenon that occurs whenever the Right Endpoint Rule is used on a decreasing curve).
(c) Write a mathematical statement that expresses the area under the curve from \( x = 0 \) to \( x = 1 \) as a limit, again using the Right Endpoint Rule. Explain any notation you use. (You should not evaluate this limit.)

\[
\text{Area} = \lim_{n \to \infty} \left( \frac{1}{n} \cdot \left( f\left( \frac{1}{n} \right) + f\left( \frac{2}{n} \right) + \cdots + f\left( \frac{n}{n} \right) \right) \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \cdot \left( \sqrt{1-\left(\frac{1}{n}\right)^2} + \sqrt{1-\left(\frac{2}{n}\right)^2} + \cdots + \sqrt{1-\left(\frac{n}{n}\right)^2} \right)
\]

Any of these expressions is acceptable; Sigma notation optional.

(d) Find the value of the definite integral \( \int_{0}^{1} \sqrt{1-x^2} \, dx \). Justify your answer. (Hint: it’s wise to avoid using the Evaluation Theorem.)

\[
\int_{0}^{1} \sqrt{1-x^2} \, dx = \text{Area under curve}
\]

\[
= \text{Area of quarter-circle of radius 1}
\]

\[
= \frac{1}{4} \cdot \pi \cdot 1^2 = \frac{\pi}{4}.
\]

[This integral can be done via the Evaluation Theorem, but a somewhat cumbersome \( u \)-substitution (or else a cumbersome manipulation using integration by parts) is required; see Stewart Section 5.7 if you’re interested.]
4. (6 points) Each of the following expressions involves an integral. When the expression is evaluated, the result is either a number or a symbolic quantity involving one or more variables. For each, say whether the expression can be evaluated to:

(I) a number, involving no variables,
(II) a quantity involving the variable \( x \) but not involving \( t \),
(III) a quantity involving the variable \( t \) but not involving \( x \), or
(IV) a quantity that must involve both variables \( t \), \( x \) (i.e., none of (I)-(III)).

No justification is necessary, and there is no penalty for guessing, but each line should have only one response. (Note: you needn’t actually evaluate all of these expressions.)

<table>
<thead>
<tr>
<th>Expression</th>
<th>I, II, III or IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int t^3 , dt )</td>
<td>( \text{III} )</td>
</tr>
<tr>
<td>( \int_0^1 \sqrt{t} , dt )</td>
<td>I</td>
</tr>
<tr>
<td>( \int_0^1 \sqrt{x^3 + 1} , dx )</td>
<td>I</td>
</tr>
<tr>
<td>( \int_t^x \sqrt{t^3 + 1} , dt )</td>
<td>II</td>
</tr>
<tr>
<td>( \frac{d}{dx} \int_1^z \sqrt{t^3 + 1} , dt )</td>
<td>II</td>
</tr>
<tr>
<td>( \frac{d}{dx} \int_0^1 \sqrt{t^3 + 1} , dt )</td>
<td>I</td>
</tr>
</tbody>
</table>
4. (18 points) Let \( f(x) = \int_{-3}^{x} \sqrt[3]{t^2 - 1} \, dt \).

(a) Find \( f(-3) \) and \( f'(-3) \).

\[
\begin{align*}
\frac{f(-3)}{\int_{-3}^{-3} \sqrt[3]{t^2 - 1} \, dt} &= 0 \quad \text{(limits equal)}
\end{align*}
\]

By Fundamental Theorem of Calculus,

\[
\frac{f'(x)}{f'(x) = \frac{3}{\sqrt[3]{x^2 - 1}}} \quad \text{and so} \quad f'(-3) = \frac{3}{\sqrt[3]{(-3)^2 - 1}} = \frac{3}{\sqrt[3]{8}} = \sqrt[3]{2}.
\]

(b) Find the intervals on which \( f \) is increasing or decreasing.

As noted above (by FTC), \( f'(x) = \frac{3}{\sqrt[3]{x^2 - 1}} \). We ask: when is \( f' \)

\[
\frac{\text{positive? negative?}}{= 0}.
\]

First check where \( f'(x) \) is 0 or undefined:

\[
\begin{align*}
- f'(x) = 0 & \Rightarrow \frac{3}{\sqrt[3]{x^2 - 1}} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1. \\
- f'(x) \text{ is never undefined (cube root of a number always exists)}
\end{align*}
\]

Now check intervals:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sign of ( x^2 - 1 )</th>
<th>Sign of ( \frac{3}{\sqrt[3]{x^2 - 1}} )</th>
<th>( f ) incr/decr?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; -1 )</td>
<td>+</td>
<td>+</td>
<td>( f ) incr.</td>
</tr>
<tr>
<td>(-1 &lt; x &lt; 1)</td>
<td>( \ominus )</td>
<td>( \ominus )</td>
<td>( f ) decr.</td>
</tr>
<tr>
<td>( x &gt; 1 )</td>
<td>+</td>
<td>+</td>
<td>( f ) incr.</td>
</tr>
</tbody>
</table>

So, \( f \) is decreasing for \( -1 < x < 1 \), and increasing for \( x > 1 \), \( x < -1 \).
For easy reference, \( f(x) = \int_{-3}^{x} \sqrt{t^2 - 1} \, dt \).

(c) Find the intervals of concavity of \( f \).

Since \( f'(x) = \frac{3}{2} \sqrt{x^2 - 1} \), have \( f''(x) = \frac{1}{3} (x^2 - 1)^{-\frac{1}{2}} \cdot (2x) = \frac{2x}{3(x^2 - 1)^{\frac{1}{2}}} \).

We ask: when is \( f'' \) positive/negative? First, find where \( f'' \) is 0 or undefined.

- \( f''(x) = 0 \) when \( 2x = 0 \Rightarrow x = 0 \).
- \( f''(x) \) undefined when \( (x^2 - 1)^{\frac{1}{2}} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1 \).

Check signs of \( f''(x) \) on intervals:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sign of ( 2x )</th>
<th>Sign of ( (x^2 - 1)^{\frac{1}{2}} )</th>
<th>Sign of ( f''(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; -1 )</td>
<td>( - )</td>
<td>( + )</td>
<td>( \frac{\circ}{\circ} = \frac{\circ}{\circ} )</td>
</tr>
<tr>
<td>( -1 &lt; x &lt; 0 )</td>
<td>( - )</td>
<td>( + )</td>
<td>( \frac{\circ}{\circ} = \frac{\circ}{\circ} )</td>
</tr>
<tr>
<td>( 0 &lt; x &lt; 1 )</td>
<td>( + )</td>
<td>( + )</td>
<td>( \frac{\circ}{\circ} = \frac{\circ}{\circ} )</td>
</tr>
<tr>
<td>( x &gt; 1 )</td>
<td>( + )</td>
<td>( + )</td>
<td>( \frac{\circ}{\circ} = \frac{\circ}{\circ} )</td>
</tr>
</tbody>
</table>

Thus, \( f \) is concave down for \( x < -1 \) and \( -1 < x < 0 \).

Thus, \( f \) is concave up for \( 0 < x < 1 \) and \( x > 1 \).

(d) Using the information you found above, sketch a plausible graph of \( f \) below. Be sure to label your axes as appropriate.

(Note: it's important to have a y-scale marked in order to depict the correct slope at \( x = -3 \), but it's difficult to know the y-positions of the local extrema & inflection point, so plenty of leeway was given!)
5. (32 points) Evaluate the following integrals, showing all of your work.

(a) \( \int_4^{12} \frac{x}{\sqrt{1 + 2x}} \, dx \)

Let \( u = 1 + 2x \),

so \( du = 2\,dx \), i.e. \( \frac{du}{2} = dx \).

Also, \( x = \frac{u-1}{2} \),

and \( \begin{cases} \text{if } x=12 \text{ then } u=1+2\cdot12=25, \\ \text{if } x=4 \text{ then } u=1+2\cdot4 = 9. \end{cases} \)

\[
\int_4^{12} \frac{x}{\sqrt{1 + 2x}} \, dx = \int_9^{25} \frac{(u-1)}{\sqrt{u}} \cdot \frac{du}{2} \\
= \frac{1}{4} \int_9^{25} \frac{u-1}{\sqrt{u}} \, du \\
= \frac{1}{4} \int_9^{25} \left( \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} \right) \, du \\
= \frac{1}{4} \left( \frac{2}{3} \cdot 25^{3/2} - 2 \cdot 25^{1/2} - \left( \frac{2}{3} \cdot 9^{3/2} - 2 \cdot 9^{1/2} \right) \right) \\
= \frac{1}{4} \left( \frac{2}{3} \cdot 125 - 2 \cdot 5 - \left( \frac{2}{3} \cdot 27 - 2 \cdot 3 \right) \right) = \frac{46}{3}. \text{ (Simplifying unnecessary)}
\]

(Note: could also have solved using integration by parts.)
(b) \[ \int \frac{\ln x}{x^{1/3}} \, dx \]

Integration by parts:

\[ u = \ln x \quad du = \frac{dx}{x} \]

\[ dv = x^{-1/3} \, dx \quad v = \frac{3}{2} x^{2/3} \]

So,

\[ \int \frac{\ln x}{x^{1/3}} \, dx = \frac{3}{2} x^{2/3} \ln x - \int \frac{3}{2} x^{2/3} \frac{1}{x} \, dx \]
\[ = \frac{3}{2} x^{2/3} \ln x - \frac{3}{2} \int x^{-1/3} \, dx \]
\[ = \frac{3}{2} x^{2/3} \ln x - \frac{3}{2} \cdot \frac{3}{2} x^{2/3} + C \]
\[ = \left[ \frac{3}{2} x^{2/3} \ln x - \frac{9}{4} x^{2/3} + C \right] \]
(c) \[ \int \sqrt{t} e^{\sqrt{t}} dt \]

Substitution: let \( u = \sqrt{t} \), so \( du = \frac{1}{2\sqrt{t}} dt \).

(Also, since \( t = u^2 \), have \( dt = 2udu \).)

So,
\[
\int \sqrt{t} e^{\sqrt{t}} dt = \int u e^u (2udu) = 2\int u^2 e^u du.
\]

Now, integration by parts. Let \( f = u^2 \),
\[
g' = e^u,
\]
so:
\[
f' = 2u,
g = e^u,
\]
and
\[
\int \sqrt{t} e^{\sqrt{t}} dt = 2\int u^2 e^u du
\]
\[
= 2\left[ u^2 e^u - \int 2ue^u du \right]
\]
\[
= 2u^2 e^u - 4\int ue^u du.
\]

Need another integration by parts:
\[
h = u,
\]
so:
\[
h' = 1,
\]
\[
k' = e^u,
k = e^u,
\]
and
\[
\int \sqrt{t} e^{\sqrt{t}} dt = 2u^2 e^u - 4\int ue^u du
\]
\[
= 2u^2 e^u - 4\left[ ue^u - \int e^u du \right]
\]
\[
= 2u^2 e^u - 4ue^u + 4\int e^u du
\]
\[
= 2u^2 e^u - 4ue^u + 4e^u + C
\]
\[
= 2te^{\sqrt{t}} - 4\frac{e^{\sqrt{t}}}{\sqrt{t}} + 4e^{\sqrt{t}} + C
\]
5. (20 points) Find each of the following. Give complete reasoning.

(a) \[ \int \frac{x}{1-x^2} \, dx \]

Let \( u = 1-x^2 \), so
\[ du = -2x \, dx, \text{ and} \]
\[ -\frac{1}{2} \, du = x \, dx. \]

\[ \int \frac{x}{1-x^2} \, dx = \int \frac{-\frac{1}{2} \, du}{u} = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u| + C \]
\[ = -\frac{1}{2} \ln |1-x^2| + C \]

(b) \[ \int_{3}^{5} \frac{\ln x}{x} \, dx \]

Let \( u = \ln x \), so
\[ du = \frac{1}{x} \, dx, \text{ and} \]
\[ \begin{cases} \text{if} \ x = 3, \text{ then } u = \ln 3; \\ \text{if} \ x = 5, \text{ then } u = \ln 5. \end{cases} \]

\[ \int_{3}^{5} \frac{\ln x}{x} \, dx = \int_{\ln 3}^{\ln 5} u \, du = \frac{1}{2} (u^2) \bigg|_{\ln 3}^{\ln 5} = \frac{1}{2} ((\ln 5)^2 - (\ln 3)^2) \]
7. (16 points) Suppose functions $f$ and $g$ are given by

$$f(x) = \begin{cases} 
-2 & \text{if } x < 0, \\
1 & \text{if } 0 \leq x \leq 2, \\
-1 & \text{if } x > 2,
\end{cases} \quad \text{and } g(x) = \int_{-2}^{x} f(t) \, dt.$$

(a) Sketch a graph of $f$ below; be sure to label the scale on your axes.

(b) Evaluate $g(-3)$, $g(0)$, and $g(3)$, showing your reasoning.

**Solution #1:**

$$g(-3) = \int_{-2}^{-3} f(t) \, dt = \left[ \int_{-2}^{0} f(t) \, dt \right]_{-2}^{-3} = -2 \left[ 0 - (-2) \right] = 4.$$

$$g(0) = \int_{-2}^{0} f(t) \, dt = \left[ \int_{-2}^{0} f(t) \, dt \right]_{-2}^{0} = -2 \left[ 0 - (-2) \right] = -4.$$

$$g(3) = \int_{-2}^{3} f(t) \, dt = \left[ \int_{-2}^{0} f(t) \, dt \right]_{-2}^{2} + \left[ \int_{0}^{3} f(t) \, dt \right]_{-2}^{3} = -2 \left[ 0 - (-2) \right] + (2 - 0) + (3 - (-2)) = 4 + 2 + (-1) = 5.$$

**Solution #2:** (outline)

Interpret each value of $g$ as an area (with sign) under the curve $f$, and use the sketch of $f$.

**Example:** $g(3)$ is the area under $f$ between $x = -2$ and $x = 3$, which is sketched in the diagram on the right, and so

$$g(3) = -(\text{area of first rectangle}) + (\text{area of second rectangle}) - (\text{area of third}) = -4 + 2 - 1 = -3.$$
For easy reference, here are \( f \) and \( g \) again:

\[
f(x) = \begin{cases} 
-2 & \text{if } x < 0, \\
1 & \text{if } 0 \leq x \leq 2, \\
-1 & \text{if } x > 2,
\end{cases}
\]

and \( g(x) = \int_{-2}^{x} f(t) \, dt. \)

(c) What is \( g'(4) \)? Give complete reasoning.

By the Fundamental Theorem of Calculus,

\[
g'(x) = \frac{d}{dx} \left( g(x) \right) = \frac{d}{dx} \left( \int_{-2}^{x} f(t) \, dt \right) = f(x).
\]

Thus, \( g'(4) = f(4) = -1 \).

(d) Find an explicit formula (i.e., no integral signs) that gives the value of \( g(x) \) in terms of \( x \), where \( x \) lies in the interval \( 0 \leq x \leq 2 \). Justify your answer.

**Solution #1**: Since \( g'(x) = f(x) \), we know that \( g'(x) = 1 \) for \( 0 \leq x \leq 2 \).

Thus, \( g(x) = x + C \) for values of \( x \) between 0 and 2, and for some constant \( C \). However, in part (c) it was found that \( g(0) = -4 \), which means that \( C = -4 \) and \( \boxed{g(x) = x - 4} \) is the formula for \( g \) when \( 0 \leq x \leq 2 \).

**Solution #2**: Use the formula for \( g \) above, and calculate the integral for \( 0 \leq x \leq 2 \):

\[
g(x) = \int_{-2}^{x} f(t) \, dt = \int_{-2}^{0} f(t) \, dt + \int_{0}^{x} f(t) \, dt
\]

\[
= \int_{-2}^{0} f(t) \, dt + \int_{0}^{x} 1 \, dt
\]

\[
= -2t \bigg|_{-2}^{0} + t \bigg|_{0}^{x} = -2(0 - (-2)) + (x - 0) = \boxed{x - 4}.
\]