Mathematics Department Stanford University
Math 51H Final Examination, December 9, 2013

3 Hours

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
Q.5	
Q.6	
Q.7	
Q.8	
T/40	

Name (Print Clearly): ____________________________

I understand and accept the provisions of the honor code (Signed) ____________________________

1(a) (2 points): Calculate the determinant of

$$\begin{pmatrix} 11 & 12 & 13 & 426 \\ 2001 & 2002 & 2003 & 421 \\ 2 & 1 & 0 & -419 \\ 101 & 101 & 102 & 2000 \end{pmatrix}$$

No calculators: Clearly state all column/row operations.

(b) (3 points) Find the matrix of the orthogonal projection onto the plane $V = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0\}$.

Hint: Start by finding the orthogonal projection onto the (1-dimensional) normal space $V^\perp$.

2(a) (2 points): If $u : \mathbb{R}^n \to \mathbb{R}$ is C^1 and if $\gamma : \mathbb{R} \to \mathbb{R}^n$ is also C^1, prove that the velocity vector $\Gamma'(t)$ of the curve $\Gamma(t) = \begin{pmatrix} \gamma(t) \\ u(\gamma(t)) \end{pmatrix}$ is orthogonal to the vector $\begin{pmatrix} \nabla u(\gamma(t)) \\ -1 \end{pmatrix}$ for each $t \in \mathbb{R}$.

(b) (3 points): Let e^x be defined as usual by $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$. Prove:

(i) $\lim_{x\to 0} |x|^{-p} e^{-1/x^2} = 0$ for each $p > 0$.

Note: You can of course assume, without giving the proof, the standard property $e^{u+v} = e^u e^v$ (so in particular $e^{-u} = 1/e^u$).

(ii) If $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$, find the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ of f.

Hint for (ii): Start by checking (by induction on n) that for $x \neq 0$ each derivative $f^{(n)}(x)$ has the form $p_n(1/x)e^{-1/x^2}$, where p_n is a polynomial.

3(a) (2 points): Define the term "open set" in $\mathbb{R}^n$, and prove that the intersection $U \cap V$ of 2 open sets U, V is again an open set.

3(b) (3 points): If $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ are both continuous, and if $S = \{\underline{x} \in \mathbb{R}^n : \varphi(\underline{x}) = 0\}$ is bounded, prove there is a point $\underline{a} \in S$ such that $f(\underline{x}) \leq f(\underline{a})\ \forall\, \underline{x} \in S$.

4(a) (3 points): State (without proof) the Spectral Theorem for a real symmetric $n \times n$ matrix A, and use it to prove that for a given quadratic form $H(\underline{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ ($a_{ij} = a_{ji}$ real) there is a change of coordinates $\underline{y} = Q^{\mathrm{T}} \underline{x}$ with Q orthogonal (i.e. $Q^T Q = QQ^T = I$) such that the quadratic form $H(\underline{x})$ is transformed to an expression of the form $\sum_{j=1}^{n} \lambda_j y_j^2$ for suitable real $\lambda_1, \ldots, \lambda_n$.

(b) (2 points): Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

5(a) (2 points): Give the "(ε, δ) definition" of continuity of a function $f : (a, b) \to \mathbb{R}$ at a point $c \in (a, b)$, and using the definition prove that if $f : (0, 1) \to \mathbb{R}$ is continuous at a point $c \in (0, 1)$ and if $f(c) = 1$ then there is $\delta > 0$ such that $f(x) > \frac{1}{2}$ for all $x \in (c - \delta, c + \delta)$.

5(b) (3 points): Prove that the function $f(x, y) = 1 - 2x - y + 4x^2 + 4xy + 2y^2 + 3xy \sin xy$ has a critical point at $(x, y) = (\frac{1}{4}, 0)$ and that f has a strict local minimum there.

6(a) (2 points): Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $\underline{v}_1 = (1,1,0,0)^{\mathrm{T}}, \underline{v}_2 = (0,1,1,0)^{\mathrm{T}}, \underline{v}_3 = (0,0,1,1)^{\mathrm{T}}$.

(b) (3 points): Find the set of all solutions of the inhomogeneous system $A\underline{x} = \underline{y}$ where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix}$$

(Give your answer as an affine space.)

7(a) (2 points): Find all eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

7(b) (3 points): Show that the system of two non-linear equations

$$(x^2 - y^2)y + 7x = 1$$
$$(x^2 - y^2)x + 5y = 1$$

has a solution (x, y) with $x^2 + y^2 < 1$.

Hint: Define $f(x, y) = \big(\frac{1}{7}\big(1 - (x^2 - y^2)y\big), \frac{1}{5}\big(1 - (x^2 - y^2)x\big)\big)$ and start by proving that f is a contraction mapping $D \to D$, where $D = \{(x, y) : x^2 + y^2 \le 1\}$.

8(a) (2 points): Let A be an $n \times n$ real matrix (a_{ij}). Define the adjoint matrix $\operatorname{adj} A$ and give the proof that $A \operatorname{adj} A = (\det A)I$.

8(b) (3 points): Show that $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 + z^2 = 1\}$ is a 2-dimensional C^1 manifold and find a point $\underset{\sim}{a} \in S$ at which the function $f(x, y, z) = xyz$ takes its maximum.

Note: You should begin by discussing the existence of such a point $\underset{\sim}{a} \in S$.

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