

Mathematics Department Stanford University
Math 51H Final Examination, December 8, 2014

2 Hours

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Question 6b is extra credit only! Work on it only if you are done with the other problems!

Q.1	_____
Q.2	_____
Q.3	_____
Q.4	_____
Q.5	_____
Q.6a	_____
T/35	_____
Q.6b	_____

Name (Print Clearly): _____

I understand and accept the provisions of the honor code (Signed) _____

1(a) (3 points): Find an orthonormal basis for the subspace of \mathbb{R}^4 spanned by the vectors $(1, 0, 2, 0)^T$, $(1, 0, 0, 3)^T$, $(0, 2, 0, 1)^T$, and write down an explicit formula (involving numbers and matrix operations only) for the matrix of the orthogonal projection to this subspace (but you do not need to compute it).

(b) (2 points) Suppose A is an $m \times n$ matrix. Prove that the dimensions of $C(A)$ and $C(A^T)$ are the same.

2(a) (4 points): For $(x, y, z) \in \mathbb{R}^3$, let $f(x, y, z) = \frac{16}{3}x^2 + z$ and let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^4 + z^6 = 1\}$. (i) Show that $f|_S$ attains a maximum and a minimum value. (ii) Find a point where each of these is attained.

(b) (3 points): Find the determinant and the inverse of the matrix

$$A = \begin{pmatrix} 3 & 6 & -9 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

3(a) (3 points): Suppose $U \subset \mathbb{R}^n$ is open, $\underline{x}_0 \in U$. (i) State the definition of a map $f : U \rightarrow \mathbb{R}^m$ being differentiable at \underline{x}_0 . (ii) Show that if $f : U \rightarrow \mathbb{R}^m$ is differentiable at \underline{x}_0 then it is continuous at \underline{x}_0 .

3(b) (4 points): Show the intermediate value theorem: if $f : [a, b] \rightarrow \mathbb{R}$ is continuous, $f(a) = \alpha$, $f(b) = \beta$, $\alpha < c < \beta$ then there exists $x \in (a, b)$ such that $f(x) = c$.

Hint: Consider $\inf\{z \in [a, b] : f(z) > c\}$.

4(a) (3 points): Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix. Prove that the quadratic form $\mathcal{A}(\underline{x}) = \sum_{i,j=1}^n a_{ij}x_i x_j$ is positive definite \iff all the eigenvalues of A are positive.
Hint: Spectral Theorem.

(b) (3 points): Find all critical points of the map $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f(x, y, z) = \frac{1}{5}(x^5 + y^5) + \frac{1}{3}z^3 - x - y - 4z$, and discuss whether these points are local maxima/minima for f . Justify all claims with proofs, possibly using theorems from lecture.

5(a) (3 points): Suppose that $\{M_n\}_{n=1}^{\infty}$ is a sequence with $M_n \geq 0$ for all n , and $\sum_{n=1}^{\infty} M_n$ converges. Show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in a complete normed vector space $(V, \|\cdot\|)$ (if you wish, you may take $V = \mathbb{R}^m$ with the standard norm) and $\|x_n\| \leq M_n$ for all n , then $\sum_{n=1}^{\infty} x_n$ converges in V , i.e. $\lim_{k \rightarrow \infty} \sum_{n=1}^k x_n$ exists. (This is the Weierstrass M -test.)

(b) (4 points): Suppose that A is an $m \times m$ matrix and it satisfies $\|A\| < 1$. Show that $(I - A)^{-1} = I + \sum_{n=1}^{\infty} A^n$. (This includes showing that the right hand side converges!)

Hint: Recall that $\|AB\| \leq \|A\|\|B\|$.

6(a) (3 points): Let I be the identity operator on \mathbb{R}^n . Show that if $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear then the statements ‘ $AB = I$ for some $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ linear’ and ‘ $CA = I$ for some $C : \mathbb{R}^n \rightarrow \mathbb{R}^n$ linear’ are equivalent, and necessarily $B = C$ in either case.

(b) (5 points extra credit only): Consider the set $O(n)$ of $n \times n$ matrices A with real entries such that $A^T A = I$. (i) Show that $O(n)$ is a group under matrix multiplication. (ii) Show that there is an open set V in \mathbb{R}^{n^2} containing I such that $O(n) \cap V$ is a C^1 submanifold of \mathbb{R}^{n^2} . What is its dimension?

Note for (ii): In fact, $O(n)$ is a C^1 , and indeed C^∞ , submanifold of \mathbb{R}^{n^2} , but you do not need to show it.

Hint for (i): Recall that matrix multiplication is associative, and I is a unit for this, so you need to show that $O(n)$ is closed under multiplication, inverses and contains I . Hint for (ii): Consider the map $A \mapsto \{(A^T A)_{ij} : i \leq j\}$ into the above diagonal (including diagonal) entries of the symmetric matrix $A^T A$.

