Mathematics Department Stanford University

Math 51H Second Mid-Term, November 12, 2013

75 MINUTES

Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
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T/25	

Name (Print Clearly): ______________________

I understand and accept the provisions of the honor code (Signed) ______________________

1(a) (3 points) State the chain rule for the composite function $g \circ f$, where $f : U \to V$ and $g : V \to \mathbb{R}^p$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open. Using the chain rule, or otherwise, prove that if $g : \mathbb{R}^n \to \mathbb{R}$ is differentiable on $\mathbb{R}^n$, if $\underline{a}, \underline{b} \in \mathbb{R}^n$, and if $h(t) = g(\underline{a} + t\underline{b})$ for $t \in \mathbb{R}$, then $h'(0)$ exists, and find its value in terms of the components $b_1, \ldots, b_n$ of $\underline{b}$ and the partial derivatives of g at $\underline{a}$.

Solution: The chain rule says that if f is differentiable at $\underline{a} \in U$ and if g is differentiable at $f(\underline{a}) \in V$, then the composite $g \circ f$ is differentiable at $\underline{a}$ and $D(g \circ f)(\underline{a}) = Dg(f(\underline{a}))Df(\underline{a})$ where the expression on the right is matrix multiplication of the $p \times n$ matrix $Dg(f(\underline{a}))$ and the $n \times m$ matrix $Df(\underline{a})$.

Notice that $h(t) = g \circ f(t)$ where $f : \mathbb{R} \to \mathbb{R}^n$ is defined by $f(t) = \underline{a} + t\underline{b}$. Since in this case $p = 1$ and $m = 1$, the above chain rule says $\frac{d}{dt}(g(\underline{a} + t\underline{b})) = (Dg)(\underline{a} + t\underline{b})f'(t)$, and $f'(t)$ is the constant vector $\underline{b}$; thus $\frac{d}{dt}(f(\underline{a} + t\underline{b})) = \sum_{j=1}^n (D_j g)(\underline{a} + t\underline{b})b_j$, which at $t = 0$ gives $h'(0) = \sum_{j=1}^n b_j D_j g(\underline{a})$.

1(b) (3 points.) (i) Give the definition of "U is open" and "C is closed" as applied to subsets $U, C \subset \mathbb{R}^n$, and (ii) give the proof that $\mathbb{R}^n \setminus U$ closed implies U open.

Note: In lecture we proved $\mathbb{R}^n \setminus U$ is closed $\iff$ U is open; in (ii) you are only being asked to prove "$\Rightarrow$."

Solution: U open means that for each $\underline{y} \in U$ there is a $\rho > 0$ such that $B_\rho(\underline{y}) \subset U$. C closed means that C contains all its limit points. That is if $\{\underline{x}_k\}$ is a convergent sequence and $\underline{x}_k \in C$ for each k, then $\lim \underline{x}_k \in C$.

Let $\underline{y} \in U$. We claim that there is $\rho > 0$ such that $B_\rho(\underline{y}) \subset U$. Otherwise we would have $B_{1/k}(\underline{y}) \cap (\mathbb{R}^n \setminus U) \neq \emptyset$ for each $k = 1, 2, \ldots$, and hence we could select $\underline{x}_k \in B_{1/k}(\underline{y}) \cap (\mathbb{R}^n \setminus U)$ for each $k = 1, 2, \ldots$. Then $\|\underline{x}_k - \underline{y}\| < 1/k$ for each k and hence $\underline{x}_k \to \underline{y}$, thus proving that $\underline{y}$ is a limit point of $\mathbb{R}^n \setminus U$. But $\mathbb{R}^n \setminus U$ is closed, so all limit points of $\mathbb{R}^n \setminus U$ must be in $\mathbb{R}^n \setminus U$, contradicting the fact that $\underline{y} \in U$.

2(a) (3 points): Suppose $\delta > 0$ and $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n$ are convergent power series in $(-\delta, \delta)$. Prove $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for each $x \in (-\delta, \delta)$ implies that $a_n = b_n$ for each $n = 0, 1, 2, \ldots$.

Hint: Since we can take $c_n = a_n - b_n$, it suffices to prove $\sum_{n=0}^{\infty} c_n x^n = 0 \, \forall \, x \in (-\delta, \delta) \Rightarrow c_n = 0 \, \forall \, n = 0, 1, 2, \ldots$.

Solution: Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < \delta$. By a theorem of lecture we have $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n x^{n-k}$ for $|x| < \delta$, so $f^{(k)}(0) = k! c_k$. But of course $f^{(k)}(0) = 0$ because we are given $f \equiv 0$, so in fact $c_k = 0$ for each k.

Alternative Method for 2(a): Setting $x = 0$ in the identity $\sum_{n=0}^{\infty} c_n x^n = 0$ we get $c_0 = 0$ hence $x \sum_{n=1}^{\infty} c_n x^{n-1} = 0$ for $|x| < \delta$ and so $\sum_{n=1}^{\infty} c_n x^{n-1} = 0$ for $0 < |x| < \delta$. However by a theorem of lecture power series are differentiable (hence continuous) in their interval of convergence, so $\lim_{x \to 0} \sum_{n=1}^{\infty} c_n x^{n-1} (= 0)$ is just the value of $\sum_{n=1}^{\infty} c_n x^{n-1}$ at $x = 0$, which is c_1. Hence $c_1 = 0$. Continuing in this manner (formally by induction on n) we get $c_n = 0$ for each n.

2(b) (3 points.) (i) Prove that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is AC on all of $\mathbb{R}$.

(ii) If we define $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, prove that $\exp(x+t) = (\exp x)(\exp t)$.

Hint for (ii): For fixed t let $f(x) = \exp(x+t)$ and $g(x) = (\exp t)(\exp x)$. Start by checking that $f^{(n)}(0) = g^{(n)}(0)$ for each $n = 0, 1, 2, \ldots$.

Solution: (i) If $x \neq 0$ we have $|x^{n+1}/(n+1)!|/|x^n/n!| = |x|/(n+1) \to 0$, hence by the ratio test the series is AC (hence convergent) for all x.

(ii) Using the theorem from lecture the power series is differentiable inside the interval of convergence and the derivative can be correctly computed by using termwise differentiation in the interval of convergence. Thus $\frac{d}{dx} \exp x = \sum_{n=1}^{\infty} n x^{n-1}/n! = \sum_{n=1}^{\infty} x^{n-1}/(n-1)! = \sum_{n=0}^{\infty} x^n/n! = \exp x$, and so by the chain rule $\frac{d}{dx} \exp(x+t) = \exp(x+t)$, and by repeatedly applying this we see that $g^{(n)}(x) = \exp t \exp x$ and $f^{(n)}(x) = \exp(x+t)$. In particular $f^{(n)}(0) = g^{(n)}(0) = \exp t$, so f and g have the same Taylor series with base-point zero. Now exp differentiable on $\mathbb{R}$ hence continuous hence $|\exp x|$ bounded on each closed interval $|x| \leq R$. Let $M = \sup_{|s| < |t| + R} |\exp s|$. Then $R^n \sup_{|x| < R} |f^{(n)}(x)|/n! = R^n \sup_{|x| < R} |\exp(x+t)|/n! \leq M R^n/n!$. Since $R^n/n! \to 0$ as $n \to \infty$, this shows that there for each $R > 0$ there is a C such that $R^n \sup_{|x| < R} |f^{(n)}(x)|/n! \leq C$ for all $n = 0, 1, 2, \ldots$, and hence by a theorem of lecture the Taylor series of f converges to $f(x) = \exp(x+t)$. Since by definition of exp the Taylor series of $g(x)$ converges to $(\exp x)(\exp t)$ and we have shown that the 2 Taylor series coincide, this shows that $\exp(x+t) = \exp x \, \exp t$ as required.

3(a) (4 points.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{3}y^3 + xy + x^2$. Find the critical points (i.e. points where $\nabla_{\mathbb{R}^2} f = \underline{0}$) of f, and state whether each critical point is a local max, local min or neither. Make sure you justify all claims you make in your argument, either with a proof or by quoting the appropriate theorem from lecture.

Solution: $\nabla f(x, y) = \begin{pmatrix} y + 2x \\ y^2 + x \end{pmatrix} = \underline{0} \iff x = -y/2 = -y^2 \iff (x, y) = (0, 0)$ or $(-1/4, 1/2)$. Thus there are two critical points $(0, 0)^T, (-1/4, 1/2)^T$. The Hessian matrix $(D_iD_jf(x, y))$ is $\begin{pmatrix} 2 & 1 \\ 1 & 2y \end{pmatrix}$ which at $(0, 0)^T$ is $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, so the Hessian quadratic form is $2y_1^2 + 2y_1y_2$ which is indefinite (e.g. it is negative at $y_1 = 1, y_2 = -2$ and positive at $y_1 = 1, y_2 = 0$), so from lecture we know that f takes neither a local max nor a local min at $(0, 0)$. At the critical point $(-1/4, 1/2)^T$ the Hessian matrix is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, so the Hessian quadratic form is $2y_1^2+2y_1y_2+y_2^2 = y_1^2+(y_1+y_2)^2$ which is clearly positive definite, so from lecture the critical point $(-1/4, 1/2)^T)$ is a strict local min for f.

3(b) (3 points.) Give the definition of "length of a curve $\gamma : [a, b] \to \mathbb{R}^n$." Using any theorem from lecture that you need, find the length of γ in case $n = 2$ and $\gamma(t) = (\sin t^2, \cos t^2), t \in [0, 2]$.

Solution: For any partition $\mathcal{P} : a = t_0 < t_2 < \cdots < t_N = b$ of $[a, b]$, we define $\ell(\gamma, \mathcal{P})$ ("the length of the polygonal approximation determined by $\mathcal{P}$") by $\ell(\gamma, \mathcal{P}) = \sum_{j=1}^N \|\gamma(t_j) - \gamma(t_{j-1})\|$, and then $\ell(\gamma) = \sup\{\ell(\gamma, \mathcal{P}) : \mathcal{P}$ is any partition of $[a, b]\}$ if the set on the right is bounded, otherwise we say the length is ∞.

The given curve is C^1, so from lecture its length is given by $\int_0^2 \|\gamma'(t)\|\, dt = \int_0^2 2t\, dt = t^2|_0^2 = 4$.

4(a) (3 points.) Prove that $M = \{\underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3^2 = 1 + x_1^2 + x_2^2\}$ is a C^1 manifold, and find the tangent space $T_{\underset{\sim}{a}}M$ at the point $\underset{\sim}{a} = (2, 2, -3)$.

Note: You should give a basis for the tangent space.

Solution: Let $g_1(x_1, x_2) = \sqrt{1 + x_1^2 + x_2^2}$ and $g_2(x_1, x_2) = -\sqrt{1 + x_1^2 + x_2^2}$ (so that g_1, g_2 are both C^∞ on all of $\mathbb{R}^2$ with $g_1 \geq 1$ and $g_2 \leq -1$). Observe that then $\underset{\sim}{a} \in \operatorname{graph} g_1$ and $\underset{\sim}{b} \in \operatorname{graph} g_2 \Rightarrow \|\underset{\sim}{a} - \underset{\sim}{b}\| \geq |a_3 - b_3| \geq 2$, so any open ball $B_\delta(\underset{\sim}{a})$ with $\underset{\sim}{a} \in \operatorname{graph} g_1$ and $\delta < 2$ does not intersect $\operatorname{graph} g_2$. Let $G_1(x_1, x_2)$ be the graph map of g_1, so that $G_1(x_1, x_2) = (x_1, x_2, g_1(x_1, x_2))$. Then for $\underset{\sim}{a}, \delta$ as above we take $\underset{\sim}{a}_0 \in \mathbb{R}^2$ with $G_1(\underset{\sim}{a}_0) = \alpha$ and then have $B_\delta(\underset{\sim}{a}) \cap M = B_\delta(\underset{\sim}{a}) \cap G_1(\mathbb{R}^2)$, and we note that $B_\delta(\underset{\sim}{a}) \cap G_1(\mathbb{R}^2) = G_1(U)$, where $U = \{(x_1, x_2) \in \mathbb{R}^2 : \|G_1(\underset{\sim}{x}) - G(\underset{\sim}{a}_0)\| < \delta\}$. Evidently U is an open set because $\underset{\sim}{y} \in U \Rightarrow \|G(\underset{\sim}{y}) - G(\underset{\sim}{a}_0)\| < \delta$ and by continuity of G_1 at $\underset{\sim}{y}$ there is $\eta > 0$ such the $\|\underset{\sim}{x} - \underset{\sim}{y}\| < \eta \Rightarrow \|G(\underset{\sim}{x}) - G(\underset{\sim}{y})\| < \delta - \|G(\underset{\sim}{y}) - G(\underset{\sim}{a}_0)\|$ and so $\|G(\underset{\sim}{x}) - G(\underset{\sim}{a}_0)\| \leq \|G(\underset{\sim}{x}) - G(\underset{\sim}{y})\| + \|G(\underset{\sim}{y}) - G_{\underset{\sim}{a}_0}\| < \delta$. Thus we have checked the definition of C^1 manifold at each point in $G_1(\mathbb{R}^2)$. Similarly we check at each point of $G_2(\mathbb{R}^2)$. Thus M is a manifold.

Now the point $\underset{\sim}{a} = (2, 2, -3) \in G_2(\mathbb{R}^2)$ (because $-3 = g_2(2, 2)$). From lecture a basis for the tangent space $T_{\underset{\sim}{a}}M$ is $D_1G_2(2, 2), D_2G(2, 2) = (1, 0, -2/3), (0, 1, -2/3)$.

Alternative method to check that M is a manifold: As proved in HW 7, Problem 5, solutions (applied to the present case when $k = 2$ and $n = 3$), M is a C^1 manifold if for each $\underset{\sim}{a} \in M$ there is an open set W containing $\underset{\sim}{a}$ and a C^1 map $g : U \to \mathbb{R}$ with U open in $\mathbb{R}^2$ and $W \cap M = G(U)$, where G is the graph map of g. In this case $M = (M \cap \{\underset{\sim}{x} : x_3 > 0\}) \cup (M \cap \{\underset{\sim}{x} : x_3 < 0\})$ and $M \cap \{\underset{\sim}{x} : x_3 > 0\} = G_1(\mathbb{R}^2)$ and $M \cap \{\underset{\sim}{x} : x_3 < 0\} = G_2(\mathbb{R}^2)$ so every point of M satisfies the above criterion either with $W = \{\underset{\sim}{x} : x_3 > 0\}$ and $U = \mathbb{R}^2$ and $g = g_1$ or with $W = \{\underset{\sim}{x} : x_3 < 0\}$ and $U = \mathbb{R}^2$ and $g = g_2$.

Second alternative method to check that M is a manifold: The given M is the zero set of the function $g(\underset{\sim}{x}) = x_3^2 - x_1^2 - x_2^2 - 1$ and the gradient $\nabla_{\mathbb{R}^3} g(\underset{\sim}{x})$ is $2(-x_1, -x_2, x_3)^T$ which is non-zero at each point of M (because $|x_3| \geq 1$ on M), so by the first part of the Lagrange Multiplier theorem of lecture we know that M is a 2-dimensional C^1 manifold.

Note: This method should ideally not really be used at this stage, since we deferred the proof of the first part of the Lagrange multiplier theorem (in fact that will not be proved until the last week of the quarter).

4(b) (3 points.) (i) If M is a k-dimensional C^1 submanifold of $\mathbb{R}^n$ ($n \geq 2$ and $1 \leq k \leq n - 1$ given), and $f : W \to \mathbb{R}$ is C^1 with W open, $W \supset M$, give the definition of "the tangential gradient $\nabla_M f$" and "a critical point of $f|M$." (ii) In the special case when $M = S^{n-1}$ (so $k = n - 1$ and f is C^1 on an open set $W \supset S^{n-1}$) prove that $f|S^{n-1}$ has at least two distinct critical points $\underset{\sim}{a}, \underset{\sim}{b} \in S^{n-1}$.

Solution (i): For $\underset{\sim}{a} \in M$, $\nabla_M f(\underset{\sim}{a})$ is defined to be $P_{T_{\underset{\sim}{a}}M}(\nabla_{\mathbb{R}^n} f(\underset{\sim}{a}))$, where $P_{T_{\underset{\sim}{a}}}M$ denotes the orthogonal projection of $\mathbb{R}^n$ onto $T_{\underset{\sim}{a}}M$. $\underset{\sim}{a} \in M$ is a critical point of $f|M$ if $\nabla_M f(\underset{\sim}{a}) = \underset{\sim}{0}$.

Solution (ii): S^{n-1} is a bounded closed subset of $\mathbb{R}^n$ hence by a result proved in lecture/homework, the continuous function $f|S^{n-1} : S^{n-1} \to \mathbb{R}$ attains a max and a min at points $\underset{\sim}{a}, \underset{\sim}{b} \in S^{n-1}$ and by another result of lecture the points $\underset{\sim}{a}, \underset{\sim}{b}$ are critical points of $f|M$. Finally $\underset{\sim}{a}, \underset{\sim}{b}$ are distinct unless the max and min values coincide, but in that case f is constant on S^{n-1} so all points of S^{n-1} are critical points of $f|S^{n-1}$ in that case.