Mathematics Department Stanford University
Math 51H Second Mid-Term, November 11, 2014

Solutions

Unless otherwise indicated, you can use results covered in lecture, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ____________________________

I understand and accept the provisions of the honor code (Signed) ____________________________

1(a) (3 points.) (i) Give the definition of "U is open" and "C is closed" as applied to subsets $U, C \subset \mathbb{R}^n$, and (ii) give the proof that $\mathbb{R}^n \setminus C$ open implies C closed.

Note: In lecture we proved $\mathbb{R}^n \setminus C$ is open $\iff$ C is closed; in (ii) you are only being asked to give the proof of "$\Rightarrow$."

Solution: U open means that for each $y \in U$ there is a $\rho > 0$ such that $B_\rho(y) \subset U$. C closed means that C contains all its limit points. That is if $\{\underline{x}_k\}$ is a convergent sequence in $\mathbb{R}^n$ and $\underline{x}_k \in C$ for each k, then $\lim \underline{x}_k \in C$.

Suppose $\{\underline{x}_k\}$ is a convergent sequence, $\underline{x}_k \in C$ for all k, $\underline{x} = \lim \underline{x}_k$, and $\mathbb{R}^n \setminus C$ is open. Suppose for the sake of contradiction that $\underline{x} \notin C$, i.e. $\underline{x} \in \mathbb{R}^n \setminus C$. Since the latter set is open, there exists $\delta > 0$ such that $B_\delta(\underline{x}) \subset \mathbb{R}^n \setminus C$. By the definition of convergence, there exists N such that $k \geq N$ implies $\|\underline{x}_k - \underline{x}\| < \delta$. Thus, $\underline{x}_N \in B_\delta(\underline{x}) \subset \mathbb{R}^n \setminus C$, contradicting $\underline{x}_N \in C$. Thus proves that $\underline{x} \in C$, i.e. C contains all of its limit points.

1(b) (4 points) (i) Give the definition of $f : \mathbb{R}^n \to \mathbb{R}^k$ being continuous, and (ii) show that if $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous and $U \subset \mathbb{R}^k$ is open, $C \subset \mathbb{R}^k$ is closed then $f^{-1}(U) = \{x : \ f(x) \in U\}$ is open and $f^{-1}(C) = \{x : \ f(x) \in C\}$ is closed.

Solution: (i) f is continuous if for all $\underline{a} \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\underline{x} - \underline{a}\| < \delta$ implies $\|f(\underline{x}) - f(\underline{a})\| < \varepsilon$.

(ii) Either of the two conclusions follows from the other as e.g. $\mathbb{R}^n \setminus f^{-1}(U) = f^{-1}(\mathbb{R}^k \setminus U)$, i.e. the complement of $f^{-1}(U)$ is $f^{-1}(C)$ with $C = \mathbb{R}^k \setminus U$, so if the 'closed' claim is shown, the 'open' one follows as a set is open if and only if its complement is closed. However, we proceed directly instead. To see the openness claim, suppose U is open, and $\underline{a} \in f^{-1}(U)$, i.e. $f(\underline{a}) \in U$. As U is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(\underline{a})) \subset U$, i.e. if $y \in \mathbb{R}^k$ with $\|y - f(\underline{a})\| < \varepsilon$ then $y \in U$. By the definition of continuity there exists $\delta > 0$ such that $\|\underline{x} - \underline{a}\| < \delta$ implies $\|f(\underline{x}) - f(\underline{a})\| < \varepsilon$, so if $\underline{x} \in B_\delta(\underline{a})$ then $\|f(\underline{x}) - f(\underline{a})\| < \varepsilon$ and so $f(\underline{x}) \in U$, i.e. $\underline{x} \in f^{-1}(U)$. This shows that $B_\delta(\underline{a}) \subset f^{-1}(U)$, so $f^{-1}(U)$ is open. To see the closedness claim, suppose that C is closed and $\{\underline{x}_j\}$ is a sequence in $f^{-1}(C)$ (i.e. $f(\underline{x}_j) \in C$) converging to some $\underline{x} \in \mathbb{R}^n$. Since f is continuous, thus sequentially continuous, $\lim f(\underline{x}_j) = f(\underline{x})$, so $\{f(\underline{x}_j)\}$ is a convergent sequence of points in C, with limit $f(\underline{x})$. Since C is closed, $f(\underline{x}) \in C$, so $\underline{x} \in f^{-1}(C)$. Thus, $f^{-1}(C)$ contains its limit points, so it is closed.

2(a) (3 points.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \frac{1}{5}(x^5 + y^5) + \frac{1}{3}x^3 - 2x - y$. Find all the critical points (i.e. points where $\nabla_{\mathbb{R}^n} f = 0$) of f, and discuss whether these points are local max/min for f. Justify all claims either with proof or by using a theorem from lecture.

Solution: $Df(x,y) = (x^4+x^2-2, y^4-1) = ((x^2-1)(x^2+2), (y^2+1)(y^2-1)) = ((x-1)(x+1)(x^2+2), (y-1)(y+1)(y^2+1))$, so there are 4 critical points $(1,1), (-1,-1), (1,-1), (-1,1)$. The Hessian matrix at (x,y) is $\begin{pmatrix} 4x^3+2x & 0 \\ 0 & 4y^3 \end{pmatrix}$ which gives positive definite quadratic form $6\lambda^2 + 4\mu^2$ at $(1,1)$ and negative definite quadratic form $-6\lambda^2 - 4\mu^2$ at $(-1,-1)$. Hence by the Second Derivative test from lecture (applicable because f is C^2, in fact C^∞), we see that f has a a local minimum at $(1,1)$ and a local maximum at $(-1,-1)$. At the point $(-1,1)$ the Hessian quadratic form is $-6\lambda^2 + 4\mu^2$ which changes sign (has positive max on S^1 and a negative min on S^1), and hence, as we proved in lecture/section, it is neither a local max nor a lcoal min for f. Similarly the point $(1,-1)$ is neither a local max nor a local min for f.

2(b) (2 points.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{1+x^2+y^2}$. Find the tangent space of the graph of f at $(2,2,3) \in \mathbb{R}^3$.

Solution: From lecture/homework the tangent space is $\text{Span}\{D_1G(\underline{0}), D_2G(\underline{0})\}$, where G is the graph map $G(x,y) = (x, y, \sqrt{1+x^2+y^2})^{\mathrm{T}}$. Thus $D_1G(x,y) = (1, 0, \frac{x}{\sqrt{1+x^2+y^2}})^{\mathrm{T}}$ and $D_2G(x,y) = (0, 1, \frac{y}{\sqrt{1+x^2+y^2}})^{\mathrm{T}}$, and hence the tangent space is $\text{Span}\{(1,0,\frac{2}{3})^{\mathrm{T}}, (0,1,\frac{2}{3})^{\mathrm{T}}\}$.

3(a) (3 points): (i) State the definition of "$\sum_{n=0}^{\infty} a_n$ converges," resp. "$\sum_{n=0}^{\infty} a_n$ converges absolutely," and (ii) show that if $\sum_{n=0}^{\infty} a_n c^n$ converges then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $x \in \mathbb{R}$ with $|x| < |c|$.

Solution: (i) $\sum_{n=1}^{\infty} a_n$ convergent means that the sequence of partial sums $\{s_n\}_{n=1,2,\ldots}$ is convergent, and in this case we say $s = \lim s_n$ is "the sum of the series" (and we write $s = \sum_{n=1}^{\infty} a_n$). $\sum_{n=1}^{\infty} a_n$ absolutely convergent means that $\sum_{n=1}^{\infty} |a_n|$ is convergent. (ii) We may assume $c \neq 0$ since otherwise the conclusion is empty. Suppose $\sum_{n=0}^{\infty} a_n c^n$ converges. Since for any convergent series the terms converge to 0, $\lim a_n c^n = 0$, so as every convergent sequence is bounded, there exists $M > 0$ such that $|a_n c^n| \leq M$ for all n. Then $|a_n x^n| = |a_n c^n||x/c|^n \leq M|x/c|^n$. Now, the series $\sum_{n=0}^{\infty} M|x/c|^n$ is a convergent geometric series since $|x/c| < 1$, so its partial sums are bounded (as they converge to the actual sum of the series). Correspondingly, the partial sums of $\sum_{n=0}^{\infty} |a_n x^n|$ are also bounded: $\sum_{n=0}^{N} |a_n x^n| \leq \sum_{n=0}^{N} M|x/c|^n$. Since a series with non-negative terms converges if and only if its partial sums are bounded, $\sum_{n=0}^{\infty} |a_n x^n|$ converges, i.e. $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

3(b) (3 points) If $\cos x, \sin x$ are defined by $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ and $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$, prove, for all $x \in \mathbb{R}$, $\frac{d}{dx} \cos x = -\sin x$, $\frac{d}{dx} \sin x = \cos x$, and $\sin^2 x + \cos^2 x = 1$.

Solution: First note that the series are convergent for all $x \in \mathbb{R}$ (e.g. by the ratio test), and hence by a theorem of lecture the series give C^1 functions which can be differentiated simply by taking the termwise differentiated series. Thus

$$\begin{aligned}
\tfrac{d}{dx} \cos x &= \sum_{k=1}^{\infty} (-1)^k 2k x^{2k-1}/(2k)! \\
&= -\sum_{k=1}^{\infty} (-1)^{k-1} x^{2k-1}/(2k-1)! = -\sum_{k=0}^{\infty} (-1)^k x^{2k+1}/(2k+1)! = \sin x \\
\tfrac{d}{dx} \sin x &= \sum_{k=0}^{\infty} (-1)^k (2k+1) x^{2k}/(2k+1)! \\
&= \sum_{k=0}^{\infty} (-1)^k x^{2k}/(2k)! = \cos x.
\end{aligned}$$

and then $\frac{d}{dx}(\sin^2 x + \cos^2 x) = 2\sin x \cos x - 2\cos x \sin x = 0$, so that $\sin^2 x + \cos^2 x$ is a constant C on all of $\mathbb{R}$. However $\cos 0 = 1$ and $\sin 0 = 0$, so $C = 1$.

4(a) (4 points.) (i) Give the definition of a curve $\gamma : [a,b] \to \mathbb{R}^n$ having finite length, and for curves of finite length state the definition of the "length of a curve $\gamma : [a,b] \to \mathbb{R}^n$." (ii) Show that if $\gamma : [a,b] \to \mathbb{R}^n$ has the property that $\gamma|_{(a,b]}$ is C^1 and $\lim_{c\to a} \int_c^b \|\gamma'(t)\|\, dt$ exists then γ has finite length, equal to $\lim_{c\to a} \int_c^b \|\gamma'(t)\|\, dt$.

Hint: Any curve is continuous by definition. Use this, and the definition of length together with the theorem from lecture for C^1 curves.

Solution: (i) A curve (a continuous map) $\gamma : [a,b] \to \mathbb{R}^n$ has finite length if the set $\{\ell(\gamma,\mathcal{P}) : \mathcal{P} \text{ partition of } [a,b]\}$ is bounded above, in which case $\ell(\gamma)$ is the supremum of this set. Here $\ell(\gamma,\mathcal{P}) = \sum_{j=1}^N \|\gamma(t_j) - \gamma(t_{j-1})\|$, where $\mathcal{P}$ is the partition $a = t_0 < t_1 < \ldots < t_N = b$.

(ii) For $c > a$, $\gamma|_{[c,b]}$ is C^1 by assumption, so by the theorem from class, it is finite length with $\ell(\gamma|_{[c,b]}) = \int_c^b \|\gamma'(t)\|\, dt$. In particular, for any partition $\mathcal{P}'$ of $[c,b]$, $\ell(\gamma|_{[c,b]},\mathcal{P}') \le \int_c^b \|\gamma'(t)\|\, dt$. Now, if $\mathcal{P}$ is any partition $a = t_0 < t_1 < \ldots < t_N = b$ of $[a,b]$, let $c = t_1$, so $c = t_1 < t_2 < \ldots < t_N = b$ is a partition $\mathcal{P}'$ of $[c,b]$, and so $\ell(\gamma,\mathcal{P}) = \|\gamma(c) - \gamma(a)\| + \ell(\gamma,\mathcal{P}') \le \|\gamma(c) - \gamma(a)\| + \int_c^b \|\gamma'(t)\|\, dt$. Since γ is continuous on $[a,b]$, so is the function $f(t) = \|\gamma(t) - \gamma(a)\|$ (being the composite of continuous functions); since $[a,b]$ is compact, f is bounded, say $f(t) \le M$ for all $t \in [a,b]$. Moreover, as $S(\tau) = \int_\tau^b \|\gamma'(t)\|,\, dt$ is a decreasing function of τ (as the integrand is non-negative),

$$\ell = \lim_{\tau\to a} S(c),$$

which exists by assumption, is actually $\sup\{S(\tau) :\ \tau \in (a,b]\}$, so is in particular $\ge \int_c^b \|\gamma'(t)\|\, dt$. Thus, $\ell(\gamma,\mathcal{P}) \le M + \ell$, proving that the length of the polygonal approximations is bounded above by $M+\ell$, and thus γ has finite length. In particular, for any $c \in (a,b)$, $\ell(\gamma) = \ell(\gamma|_{[a,c]}) + \ell(\gamma|_{[c,b]})$ as shown on the problem set, so $\ell(\gamma)$ is an upper bound for $\{\ell(\gamma|_{[c,b]}) : \underset{\sim}{c} \in (a,b]\}$, and thus $\ell(\gamma) \ge \ell$. Now, by the continuity of γ, given $\varepsilon > 0$ there is $\delta > 0$ such that $t < a+\delta$ implies $\|\gamma(t) - \gamma(a)\| < \varepsilon$. If $\mathcal{P}$ is a partition of $[a,b]$ as above, add a new division point $\sigma \in (t_0, \min(t_1,\delta))$ to obtain a new partition $\mathcal{Q}$. Then

$$\ell(\gamma,\mathcal{P}) = \|\gamma(t_1) - \gamma(t_0)\| + \ell(\gamma|_{[t_1,b]},\mathcal{P}') \le \|\gamma(\sigma) - \gamma(t_0)\| + \|\gamma(t_1) - \gamma(\sigma)\| + \ell(\gamma|_{[t_1,b]},\mathcal{P}') = \ell(\gamma,\mathcal{Q}),$$

and

$$\ell(\gamma,\mathcal{Q}) = \|\gamma(\sigma) - \gamma(t_0)\| + \ell(\gamma|_{[\sigma,b]},\mathcal{Q}') \le \varepsilon + \ell(\gamma|_{[\sigma,b]}) \le \varepsilon + \ell.$$

So for any $\varepsilon > 0$, $\ell + \varepsilon$ is an upper bound for the lengths of the polygonal approximations to γ, so $\ell(\gamma) \le \ell + \varepsilon$, i.e. as $\varepsilon > 0$ is arbitary, $\ell(\gamma) \le \ell$. Since the opposite inequality is already shown, $\ell(\gamma) = \ell$.

Note: One can streamline the argument somewhat to show the bound $\ell(\gamma,\mathcal{P}) \le \ell + \varepsilon$ directly, without showing $\ell(\gamma,\mathcal{P}) \le \ell + M$ first.

4(b) (3 points.) (i) Show that the map $\gamma : [0,1] \to \mathbb{R}^2$ given by $\gamma(0) = 0$, $\gamma(t) = (t\cos\log t, t\sin\log t)$ is continuous, C^1 on $(0,1]$, but not on $[0,1]$, and (ii) show that γ has finite length, and compute it.

Note: γ is called a logarithmic spiral. You may use the results of 4(a) even if you have not proved them.

Solution: (i) The given map is C^1 in $(0,1]$ by the chain rule, and continuous at 0 since sin, cos are bounded by 1; in fact, $0 \le t < \varepsilon$ implies $\|\gamma(t)\| = t\sqrt{\cos^2\log t + \sin^2\log t} = t < \varepsilon$. On the other hand, it is not C^1 since for $t > 0$ $\gamma'(t) = (\cos\log t - \sin\log t, \sin\log t + \cos\log t)$, and $\lim_{t\to 0}\gamma'(t)$ does not exist as is shown by taking $t_n = e^{-2n\pi}$ with $\gamma'(t_n) = (1,1)$ while for $t_n' = e^{-2n\pi+\pi}$, $\gamma'(t_n') = (-1,-1)$, with $\lim t_n \to 0$, $\lim t_n' = 0$, while if the limit existed, it would have to be equal to both of the unequal limits $\lim \gamma'(t_n)$ and $\lim \gamma'(t'_n)$. Thus, the derivative cannot be continuous at 0, so γ is not C^1. (A different way to argue is that γ is not even differentiable at 0: one needs to evaluate the difference quotients $\gamma(t)/t$, $t > 0$, and let $t \to 0$; since $\gamma(t)/t = (\cos\log t, \sin\log t)$, arguing as above shows that the limit does not exists.)

(ii) By part (a), it suffices to check that $\lim_{c\to 0} \int_c^1 \|\gamma'(t)\|\, dt$ exists. But for $t > 0$,

$$\begin{aligned}\|\gamma'(t)\| &= \sqrt{(\cos\log t - \sin\log t)^2 + (\sin\log t + \cos\log t)^2} \\ &= \sqrt{2\cos^2\log t + 2\sin^2\log t} = \sqrt{2}\end{aligned}$$

so $\ell(\gamma|_{[c,1]}) = \sqrt{2}(1-c)$, and thus $\lim_{c\to 0} \ell(\gamma|_{[c,1]}) = \sqrt{2}$, yielding that the curve has finite length, which is in fact $\sqrt{2}$.