Mathematics Department Stanford University
Math 51H Second Mid-Term, November 11, 2014

75 MINUTES

Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ____________________

I understand and accept the provisions of the honor code (Signed) ____________________

1(a) (3 points.) (i) Give the definition of "U is open" and "C is closed" as applied to subsets $U, C \subset \mathbb{R}^n$, and (ii) give the proof that $\mathbb{R}^n \setminus C$ open implies C closed.

Note: In lecture we proved $\mathbb{R}^n \setminus C$ is open $\iff$ C is closed; in (ii) you are only being asked to give the proof of "$\Rightarrow$."

1(b) (4 points) (i) Give the definition of $f : \mathbb{R}^n \to \mathbb{R}^k$ being continuous, and (ii) show that if $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous and $U \subset \mathbb{R}^k$ is open, $C \subset \mathbb{R}^k$ is closed then $f^{-1}(U) = \{x : \ f(x) \in U\}$ is open and $f^{-1}(C) = \{x : \ f(x) \in C\}$ is closed.

2(a) (3 points.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \frac{1}{5}(x^5 + y^5) + \frac{1}{3}x^3 - 2x - y$. Find all the critical points (i.e. points where $\nabla_{\mathbb{R}^n} f = 0$) of f, and discuss whether these points are local max/min for f. Justify all claims either with proof or by using a theorem from lecture.

2(b) (2 points.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{1 + x^2 + y^2}$. Find the tangent space of the graph of f at $(2,2,3) \in \mathbb{R}^3$.

3(a) (3 points): (i) State the definition of "$\sum_{n=0}^{\infty} a_n$ converges," resp. "$\sum_{n=0}^{\infty} a_n$ converges absolutely," and (ii) show that if $\sum_{n=0}^{\infty} a_n c^n$ converges then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $x \in \mathbb{R}$ with $|x| < |c|$.

3(b) (3 points) If $\cos x, \sin x$ are defined by $\cos x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$ and $\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$, prove, for all $x \in \mathbb{R}$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}\sin x = \cos x$, and $\sin^2 x + \cos^2 x = 1$.

4(a) (4 points.) (i) Give the definition of a curve $\gamma : [a, b] \to \mathbb{R}^n$ having finite length, and for curves of finite length state the definition of the "length of a curve $\gamma : [a, b] \to \mathbb{R}^n$." (ii) Show that if $\gamma : [a, b] \to \mathbb{R}^n$ has the property that $\gamma|_{(a,b]}$ is C^1 and $\lim_{c\to a} \int_c^b \|\gamma'(t)\|\, dt$ exists then γ has finite length, equal to $\lim_{c\to a} \int_c^b \|\gamma'(t)\|\, dt$.

Hint: Any curve is continuous by definition. Use this, and the definition of length together with the theorem from lecture for C^1 curves.

4(b) (3 points.) (i) Show that the map $\gamma : [0, 1] \to \mathbb{R}^2$ given by $\gamma(0) = 0$, $\gamma(t) = (t \cos \log t, t \sin \log t)$ is continuous, C^1 on $(0, 1]$, but not on $[0, 1]$, and (ii) show that γ has finite length, and compute it.

Note: γ is called a logarithmic spiral. You may use the results of 4(a) even if you have not proved them.