Mathematics Department Stanford University
Math 51H Second Mid-Term, November 10, 2015

75 MINUTES

**Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.**

**If necessary, continue solutions on backs of pages**
**Note: work sheets are provided for your convenience, but will not be graded**

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| Q.1 | |
| Q.2 | |
| Q.3 | |
| Q.4 | |
| T/25 | |

Name (Print Clearly): ____________________

I understand and accept the provisions of the honor code (Signed) ____________________

**1(a) (3 points.)** (i) Give the definition of "$U$ is open" and "$C$ is closed" as applied to subsets $U, C \subset \mathbb{R}^n$, and (ii) give the proof that if $C_1, C_2$ are closed then $C_1 \cup C_2$ is closed, and if $U_1, U_2$ are open then $U_1 \cap U_2$ is open.

Note: In (ii), at least one of the two statements should be shown directly from the definition. You may either show the other directly, or by using an appropriate theorem.

**1(b) (3 points)** (i) For $U \subset \mathbb{R}^n$ open, give the definition of $f : U \to \mathbb{R}^k$ being continuous, and (ii) show that if $f : U \to V \subset \mathbb{R}^k$ is continuous, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ are open, $g : V \to \mathbb{R}^m$ is continuous then $g \circ f$, defined by $(g \circ f)(x) = g(f(x))$, is continuous.

**2(a) (3 points.)** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{7}(x^7 + y^7) - 64x - y$. Find all the critical points (i.e. points where $\nabla_{\mathbb{R}^2} f = 0$) of $f$, and discuss whether these points are local max/min for $f$. Justify all claims either with proof or by using a theorem from lecture.

**2(b) (3 points.)** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 1 + 3x^2 + y^6 + 4(x-1)^4$. Show that $f$ is bounded below and it attains its minimum.

Note: you do not need to find where the minimum is attained. Hint: show first that if $|x| \geq 3$ or $|y| \geq 2$ then $f(x, y) \geq 65$. What is $f(0, 0)$?

**3(a) (3 points)** Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$. (i) Find its radius of convergence $\rho$. (ii) Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, $|x| < \rho$. Show that $f'(x) = \frac{1}{1-x}$ for $|x| < \rho$.

**3(b) (3 points):** (i) A sequence of functions $f_n : [a, b] \to \mathbb{R}$ converges uniformly to a function $f : [a, b] \to \mathbb{R}$ if for all $\varepsilon > 0$ there is $N \in \mathbb{N}^+$ such that $n \geq N$ implies that $\sup\{|f_n(x) - f(x)| : x \in [a, b]\} < \varepsilon$. Show that if $f_n$ are continuous and $f_n \to f$ uniformly then $f$ is continuous.

Hint: continuity of $f$ at $x$ requires given $x \in [a, b]$ and $\varepsilon > 0$ finding $\delta > 0$ with certain properties. Express $|f(y) - f(x)|$ in terms of $|f_n(y) - f_n(x)|$ and other expressions, and choose $n$ well.

**4(a) (3 points.)** (i) Give the definition of a curve $\gamma : [a, b] \to \mathbb{R}^n$ having finite length, and for curves of finite length state the definition of the "length of a curve $\gamma : [a, b] \to \mathbb{R}^n$." (ii) Show that if $\gamma : [a, b] \to \mathbb{R}^n$ has the property that there is a constant $K > 0$ such that $\|\gamma(t) - \gamma(t')\| \leq K|t - t'|$ for $t, t' \in [a, b]$ (one says $\gamma$ is Lipschitz) then $\gamma$ has finite length.

**4(b) (4 points.)** (i) Show directly (without using the corollary of the implicit function theorem that we have not proved) that the set $M = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 = z^2 + 1\}$ is a 2-dimensional $C^1$ submanifold of $\mathbb{R}^3$. (ii) Find the tangent space of $M$ at the point $(1, 1, 1)$, and give a basis for it.

Note: in fact, $M$ is a $C^\infty$ submanifold. You may use that $\sqrt{\ } : (0, \infty) \to (0, \infty)$ is $C^\infty$.