Mathematics Department Stanford University
Summary of Math 52H Material, 2009

Note: In some cases here only abbreviations of formal definitions and statements of theorems are given, and in that case a correct statement would require additional detail; All such additional detail and proofs should be known.

Riemann Integral, Volume

See `http://www.stanford.edu/class/math52h/supplements/transformation.pdf`

- Volume zero: A set A has volume zero (or "content zero") if $\forall \varepsilon > 0, \exists$ a finite collection $R_1, \cdots, R_N$ of rectangles with $A \subset \bigcup_{j=1}^N R_j$ and $\sum_{j=1}^N |R_j| < \varepsilon$.
- Lemma: A has volume zero if and only for each $\varepsilon > 0$ there is a finite collection of balls with $B_{\rho_j}(\underline{x}_j)$, $j = 1, \ldots, Q$ such that $A \subset \cup_{j=1}^Q B_{\rho_j}(\underline{x}_j)$ and $\sum_{j=1}^Q \rho_j^n < \varepsilon$.
- Boundary of a set: $\partial A = \{\underline{x} \in \mathbb{R}^n : B_\rho(\underline{x}) \cap A \neq \emptyset \text{ and } B_\rho(\underline{x}) \cap (\mathbb{R}^n \setminus A) \neq \emptyset \, \forall \rho > 0\}$. Various properties of the boundary including (i) ∂A is closed, (ii) $\partial(A \cup B) \subset \partial A \cup \partial B$, (iii) $\partial(A \cap B) \subset \partial A \cup \partial B$, and (iv) the "segment property" that if $\underline{x} \in A$ and $\underline{y} \in \mathbb{R}^n \setminus A$, then there is $t \in [0,1]$ with $t\underline{x} + (1-t)\underline{y} \in \partial A$ (which implies in particular that $\partial A \neq \emptyset$ unless $A = \emptyset$ or $A = \mathbb{R}^n$).
- Riemann integrals: Upper sum $U = \sum_{I \in \mathcal{P}} (\sup_I f)|I|$ for $f : R \to \mathbb{R}$ bounded, where $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $\mathcal{P}$ a partition of R. Lower sum is same with $\inf_I f$ in place of $\sup_I f$. Theorem that $L(f, \mathcal{P}) \le U(f, \mathcal{Q})$ for every choice of partitions $\mathcal{P}, \mathcal{Q}$. Definition that f is Riemann integrable if $\sup_{\mathcal{P}} L(f, \mathcal{P}) = \inf_{\mathcal{P}} U(f, \mathcal{P})$.
- Riemann Criterion: A bounded $f : R \to \mathbb{R}$ is Riemann integrable $\iff$ $\forall \varepsilon > 0, \exists$ a partition $\mathcal{P}$ s.t. $U(f, \mathcal{P}) < L(f, \mathcal{P}) + \varepsilon$, and in this case we have $U(f, \mathcal{P}) - \varepsilon < \int_R f < L(f, \mathcal{P}) + \varepsilon$.
- Theorem that a continuous function $f : R \to \mathbb{R}$ is Riemann integrable.
- Theorem that if $S \subset R$ has volume zero and if $f : R \to \mathbb{R}$ is bounded and is continuous at each point of $R \setminus S$, then f is Riemann integrable. Also if $g : R \to \mathbb{R}$ is bounded and $f | R \setminus S = g | R \setminus S$, then g is also Riemann integrable and $\int_R f = \int_R g$.
- Volume: the definition $\mathrm{vol}(\Omega) = \int_R \chi_\Omega$, where χ_Ω is the indicator function of Ω (and $\mathrm{vol}(\Omega)$ exists precisely when χ_Ω is Riemann integrable on a rectangle $R \supset \Omega$).
- Fact that $\mathrm{vol}(\Omega)$ exists $\iff$ for each $\varepsilon > 0$ there is a partition $\mathcal{P}$ of the rectangle $R \supset \Omega$ with $\sum_{I \in \mathcal{P}, I \cap \Omega \neq \emptyset} |I| - \sum_{I \in \mathcal{P}, I \subset \Omega} |I| < \varepsilon$ and corresponding theorem that $\mathrm{vol}(\Omega)$ exists (i.e. χ_Ω is Riemann integrable) $\iff$ $\mathrm{vol}(\partial\Omega) = 0$.
- Fubini's Thm: $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, $f : R \to \mathbb{R}$ bounded $\Rightarrow$
$$\int_R f = \int_{[a_n,b_n]} \Big(\int_{[a_1,b_1] \times \cdots \times [a_{n-1},b_{n-1}]} f(x_1, \ldots, x_{n-1}, x_n)\, dx_1 \cdots dx_{n-1} \Big) dx_n$$
provided all three integrals exist (O.K. if e.g. f is continuous on R).
- Linear Transformation of Volume Thm: If Ω is bounded and if $\mathrm{vol}(\partial\Omega) = 0$ then, for any $n \times n$ matrix A, $\mathrm{vol}(A\Omega)$ exists and $\mathrm{vol}(A\Omega) = |\det A| \mathrm{vol}(\Omega)$.
- Rough Volume Inequality: f Lipschitz ($\|f(x) - f(y)\| \le L\|x - y\| \;\forall\, x, y \in \Omega$)$\Rightarrow \mathrm{vol}(f(\Omega)) \le L^n \mathrm{vol}(\Omega)$, provided the volumes exist.
- Change of Variables Formula: U is open in $\mathbb{R}^n$, $f : U \to \mathbb{R}^n$ 1:1, C^1, $\det Df \neq 0 \,\forall x \in U$, Ω bounded, and $U \supset \Omega \cup \partial\Omega \Rightarrow \mathrm{vol}(f(\Omega))$ exists and $\int_{f(\Omega)} g = \int_\Omega g \circ f \, |\det Df|$ for any bounded continuous $g : f(\Omega) \to \mathbb{R}$. Important special case: transformation of volume

formula $\text{vol}\,(f(\Omega)) = \int_\Omega |\det Df|$, which corresponds to the choice $g \equiv 1$ in the change of variables formula.

- Alternate version of change of variables formula as on p.7 of supplement `http://www.stanford.edu/class/math52h/supplements/transformation.pdf`.
- Applications of the Change of Variables Formula, including Polar and Spherical Coordinates on various domains.

Real Analysis (Real Analysis Lectures 7,8,9)

- Theorem that AC$\Rightarrow$ convergence for complex series.
- Cauchy Product Thm: $\sum_{n=0}^\infty a_n, \sum_{n=0}^\infty b_n$ AC $\Rightarrow \sum_{n=0}^\infty c_n$ AC and $(\sum_{n=0}^\infty a_n)(\sum_{n=0}^\infty b_n) = \sum_{n=0}^\infty c_n$ where $c_n = \sum_{i=0}^n a_i b_{n-i}$ for each n.
- General vector spaces V—8 vector space axioms and examples of such spaces.
- Inner product and properties: $\langle v, w\rangle = \langle w, v\rangle$, $\langle v, w\rangle$ is linear in both v and w and $\langle v, v\rangle > 0$ unless $v = 0$. Definition $\|v\| = \sqrt{\langle v, v\rangle}$.
- Bessel's Inequality: $\sum_{n=1}^\infty c_n^2 \le \|v\|^2$.
- If f is 2π-periodic and piecewise continuous and both one-sided derivatives $\lim_{h\downarrow 0} \frac{f(x+h)-f(x_+)}{h}$ and $\lim_{h\downarrow 0} \frac{f(x-h)-f(x_-)}{h}$ exist at a point x, then the trigonometric Fourier series
$$\tfrac{a_0}{2} + \textstyle\sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx)$$
converges to $f(x)$ at that x, assuming we take $f(x) = \frac{1}{2}(f(x_+) + f(x_-))$ in case f is not continuous at x.

Differential Forms

- A 1-linear function from $\mathbb{R}^n \to \mathbb{R}$ is $\ell(\underline{v}) = \sum_{j=1}^n a_j v_j$. Special example $dx_j(\underline{v}) = v_j$; then any 1-linear ℓ can be written $\ell = \sum_{j=1}^n a_j dx_j$.
- A 1-form on U is a map $U \to \{$1-linear functions on $\mathbb{R}^n\}$. Thus $\omega|_{\underline{x}} = \sum_{j=1}^n a_j(\underline{x}) dx_j$ where a_j are given functions of $\underline{x}$ on U. ω is C^N on U means each a_j is C^N. Important special case: if $f : U \to \mathbb{R}$ is C^1 then the differential df of f, defined by $df = \sum_{j=1}^n D_j f\, dx_j$, is a 1-form on U.
- ℓ is a k-multilinear function (abbreviated k-linear here) on $\mathbb{R}^n$ if $\ell : \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k factors) $\to \mathbb{R}$ is linear in each factor (thus $\ell(\alpha\underline{v} + \beta\underline{w}, \underline{v}_2, \ldots, \underline{v}_k) = \alpha\ell(\underline{v}, \underline{v}_2, \ldots, \underline{v}_k) + \beta\ell(\underline{w}, \underline{v}_2, \ldots, \underline{v}_k)$, with a similar identity for each of the other entries).
- A k-linear function ℓ is alternating if $\ell(\underline{v}_1, \ldots, \underline{v}_i, \ldots, \underline{v}_j, \ldots, \underline{v}_k) = -\ell(\underline{v}_1, \ldots, \underline{v}_j, \ldots, \underline{v}_i, \ldots, \underline{v}_k)$ for $i \ne j$. (i.e., interchanging two entries changes the sign).
- The definition $dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\underline{v}_1, \ldots, \underline{v}_k) = \det \begin{pmatrix} v_{1i_1} & \cdots & v_{ki_1} \\ \vdots & \vdots & \vdots \\ v_{1i_k} & \cdots & v_{ki_k} \end{pmatrix}$ for any k-tuple $(i_1, \ldots, i_k)$ of integers $\in \{1, \ldots, n\}$.
- Standard Form Lemma: Any alternating k-linear function $\ell : \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k factors) $\to \mathbb{R}$ can be written $\sum_{1\le i_1<i_2<\cdots<i_k\le n} a_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $a_{i_1\cdots i_k} \in \mathbb{R}$ are unique, given by $a_{i_1\cdots i_k} = \ell(\underline{e}_{i_1}, \cdots, \underline{e}_{i_k})$. (And so the alternating k-linear functions on $\mathbb{R}^n$ are a vector space of dimension $\binom{n}{k}$ with basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$.)

- A k-form on a set $U \subset \mathbb{R}^n$ is a map $U \to$ {alternating k-linear functions}. Thus $\omega|_{\underline{x}} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} a_{i_1 \cdots i_k}(\underline{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for some choice of real-valued functions $a_{i_1,\ldots,i_k}$ on U.
- More notation: $\mathcal{I}_{k,n} \equiv \{(i_1, \cdots, i_k) : 1 \le i_1 < \cdots < i_k \le n\}$, $I = (i_1, \cdots, i_k) \in \mathcal{I}_{k,n}$, so any k-form ω can be written $\omega = \sum_{I \in \mathcal{I}_{k,n}} a_I dx_I$, where $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.
- Definition of wedge product of a k-form and an ℓ-form: $\omega = \sum_{I \in \mathcal{I}_{k,n}} a_I dx_I$, $\eta = \sum_{J \in \mathcal{I}_{l,n}} b_J dx_J$, then $\omega \wedge \eta = \sum_{I \in \mathcal{I}_{k,n}, J \in \mathcal{I}_{l,n}} a_I b_J dx_{I,J}$ where $I, J = (i_1, \ldots, i_k, j_1, \ldots, j_\ell)$. The wedge product is associative, and it is linear in each factor, $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$.
- Exterior Derivative: If ω is a C^1 k-form on U then $d\omega = \sum_{I \in \mathcal{I}_{k,n}} da_I \wedge dx_I = \sum_{I \in \mathcal{I}_{k,n}, j \in \{1,\cdots,n\}} (D_j a_I(\underline{x})) dx_j \wedge dx_I$. Thus $d\omega$ is a $(k+1)$-form on U (zero if $k \ge n$).
- Properties of exterior derivative: (i) (linear) $d(\lambda\omega_1 + \mu\omega_2) = \lambda d\omega_1 + \mu d\omega_2$, (ii) $d(f\omega) = df \wedge \omega + f d\omega$, (iii) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$, (iv) $d(d\omega) = 0$ if ω is a C^2 k-form, for $\omega, \omega_1, \omega_2$ a C^1 k-forms on U, η a C^1 ℓ-form on U and f a C^1 function on U. (Note that these also apply to the case $k = 0$—a zero C^j form is just a C^j function).
- Pullback $f^*\omega$: If $f : U \to V$ is C^1 where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, and if $\omega = \sum_{I \in \mathcal{I}_{k,m}} a_I dx_I$ is a k-form in V, then for $\underline{x} \in U$ we define $f^*\omega|_{\underline{x}} = \sum_{I \in \mathcal{I}_{k,m}} a_I(f(\underline{x})) df_I|_{\underline{x}}$ (where $df_I = df_{i_1} \wedge \cdots \wedge df_{i_k}$) $= \sum_{I \in \mathcal{I}_{k,m}, J \in \mathcal{I}_{k,n}} a_I \circ f|_{\underline{x}} \det(D_J f_I)|_{\underline{x}} \, dx_J$, which is a k-form on U.
- Basic properties of pullback: (i) (linear) $f^*(\lambda\omega_1 + \mu\omega_2) = \lambda f^*\omega_1 + \mu f^*\omega_2$, (ii) $f^*(h\omega) = h \circ f f^*\omega$, (iii) $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$, (iv) $(f \circ g)^*\omega = g^*(f^*\omega)$,
- Pullback and exterior derivative commute: $df^*\omega = f^*d\omega$ assuming $f : U \to V$ is C^2 and ω is a C^1 k-form on V.
- If $\omega = a\, dx_1 \wedge \cdots \wedge dx_n$ on an open $U \subset \Omega \cup \partial\Omega$ with Ω bounded and $\partial\Omega$ having volume zero, and if a continuous on $\Omega \cup \partial\Omega$, then we define $\int_\Omega \omega = \int_\Omega a$. Observe that (using definition of pullback) we then have $\int_\Omega f^*\omega = \int_\Omega (a \circ f) \det Df$ assuming $f : U \to V$ is C^1, U, V open in $\mathbb{R}^n$, $\omega =$ is a continuous n-form on V, and $U \supset \Omega \cup \partial\Omega$. With this notation, assuming f is in addition 1:1 and $\det Df \neq 0$, the change of variables formula can be written $\int_{f(\Omega)} \omega = \pm \int_\Omega f^*\omega$, with "+" if $\det Df > 0$ in Ω and "−" if $\det Df < 0$ in Ω

Line integrals

- Line Integral: $\gamma : [a,b] \to U$ is C^1, U open in $\mathbb{R}^n$; γ need not be 1:1, nor do we need $\gamma' \neq 0$. Definition: If ω is a continuous 1-form on U, then we define $\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega (\equiv \int_a^b \underline{a}(\gamma(t)) \cdot \gamma'(t) dt)$, which is independent of parameterization, in the sense that if $\gamma = \beta \circ \varphi$, where $\beta : [c,d] \to U$ and $\varphi : [a,b] \to \mathbb{R}$ with β, φ C^1, $\varphi' > 0$ and $\varphi([a,b]) = [c,d]$, then $\int_\gamma \omega = \int_\beta \omega$. (Proof via change of variables formula from 1-variable calculus.)
- Fundamental Thm of Calc for Line Integrals: If $\gamma : [a,b] \to U$ is continuous and piecewise C^1 and f is C^1 on U, then $\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$; note the path independence.
- U open, ω C^0 on $U \Rightarrow$ The following are equivalent:
  - ω is exact in U (i.e. $\omega = df$ for some C^1 $f : U \to \mathbb{R}$)
  - $\int_\gamma \omega$ is path independent in U (i.e. $\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$ whenever $\gamma(a) = \tilde{\gamma}(c)$ and $\gamma(b) = \tilde{\gamma}(d)$ and $\gamma : [a,b] \to U, \tilde{\gamma} : [c,d] \to U$ are both piecewise C^1 curves in U.)

- Definition of simply connected domain (U is simply connected if any for C^1 curve $\gamma : [a,b] \to U$ with $\gamma(a) = \gamma(b)$ there is a C^1 map $h : [a,b] \times [0,1] \to U$ with $h(t,0) \equiv \gamma(t)$, $h(t,1) \equiv \underline{y}$ for some $\underline{y} \in U$, and $h(a,s) = h(b,s)\ \forall\, s \in [0,1]$).) Proof (using Stokes Thm on a rectangle) that if U is simply connected then ω a closed (i.e. $d\omega = 0$) C^1 1-form on $U \Rightarrow \omega$ exact (i.e. $\omega = df$ for some C^2 function $f : U \to \mathbb{R}$).
- k-vol of k-dim parallelepiped $P \subset \mathbb{R}^n$: P spanned by $\underline{a}_1, \cdots, \underline{a}_k$ (i.e. $P = \{\sum_{j=1}^k t_j \underline{a}_j : t_1, \ldots, t_k \in [0,1]\}$) $\Rightarrow k\text{-vol}(P) = \sqrt{\det A^{\mathrm{T}} A}$, where A is the $n \times k$ matrix with columns equal to the given vectors $\underline{a}_1, \ldots, \underline{a}_k$.

Submanifolds

`http://www.stanford.edu/class/math52h/supplements/submanifold-09.pdf`

- Formal definition: M is a k-dimensional C^1 submanifold in $\mathbb{R}^n$ if there is a family of 1:1 C^1 maps $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha\in\Lambda}$ with each W_α open in $\mathbb{R}^n$ and $M \subset \cup_{\alpha\in\Lambda} W_\alpha$, and where for each $\alpha \in \Lambda$ ($\Lambda$is an index set) we have U_α open in $\mathbb{R}^k$, $D_1\varphi_\alpha(\underline{x}), \ldots, D_k\varphi_\alpha(\underline{x})$ are l.i. for each $\underline{x} \in U_\alpha$, $\varphi_\alpha(U_\alpha) = M \cap W_\alpha$, and $\varphi_\alpha^{-1} : M \cap W_\alpha \to U_\alpha$ is continuous. The φ_α are called "local coordinate charts" for M and the entire collection $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha\in\Lambda}$ is called an atlas for M.
- The fact that the above definition is equivalent to the "local graph" definition used in Math 51H.
- Fact (using inverse function theorem) that φ_α^{-1} is the restriction of a C^1 function (defined in an open set) to $M \cap W_\alpha$ (see p.2 of lecture supplement on submanifolds), and hence in particular that the transition maps are C^1.
- If $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \emptyset$ (i.e. $M \cap W_\alpha \cap W_\beta \neq \emptyset$), then $U_{\alpha\beta} = \varphi_\alpha^{-1}(M \cap W_\alpha \cap W_\beta)$ will be non-empty and we can define $\varphi_{\alpha\beta} = \varphi_\beta^{-1} \circ \varphi_\alpha : U_{\alpha\beta} \to U_{\beta\alpha}$; this is C^1 because (see above) φ_β^{-1} is the restriction to $M \cap W_\beta$ of a C^1 function. The $\varphi_{\alpha\beta}$ so defined are called "transition maps."
- Important special case: When there is just one coordinate chart $\varphi : U \to W$ (i.e. the atlas has just 1 element), so $M = \varphi(U)$. In this case we sometimes refer to M as a "k dimensional parametrized surface."
- As discussed in lecture (using an informal argument based on the fact that a C^1 map φ is well approximated near a point $\underline{x}_0$ by the affine function $\varphi(\underline{x}_0) + D\varphi(\underline{x}_0)(\underline{x} - \underline{x}_0)$) the above expression for the k-volume of a k-dimensional parallelepiped leads naturally to the following definition:

  Definition of integration of a function over a k-dimensional parametrized surface): Let $\varphi : U \to \mathbb{R}^n$ and $\Omega \subset U$ be as above and let f be a continuous real-valued function on Ω. Then we define

  $$(*) \qquad \int_{\varphi(\Omega)} f = \int_\Omega f \circ \varphi \sqrt{\det((D\varphi)^{\mathrm{T}} D\varphi)}.$$

  Remark (k-volume of a k-dimensional parametrized surface): Notice an important special case of the above definition occurs when we take $f \equiv 1$ in which case the left side is defined to be *the k-volume of $f(\Omega)$ (i.e. k-vol$(f(\Omega))$)*; thus

  $$k\text{-}\mathrm{vol}(\varphi(\Omega)) = \int_\Omega \sqrt{\det((D\varphi)^{\mathrm{T}} D\varphi)}.$$

- The fact that the above definition is independent of which map φ we use: i.e. if $\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}^n$ is a 1:1 C^1 map with $D_1\widetilde{\varphi}(\underline{x}), \dots, D_k\widetilde{\varphi}(\underline{x})$ l.i. at each $\underline{x} \in \Omega$ and if $\widetilde{\varphi}(\widetilde{U}) = \varphi(U)$ then

$$(**) \qquad \int_\Omega f \circ \varphi \sqrt{\det((D\varphi)^{\mathrm{T}} D\varphi)} = \int_{\widetilde{\Omega}} f \circ \widetilde{\varphi} \sqrt{\det((D\widetilde{\varphi})^{\mathrm{T}} D\widetilde{\varphi})}$$

with $\widetilde{\Omega} = \widetilde{\varphi}^{-1}(\varphi(\Omega))$. Notice that an extremely important example of occurs when we have two coordinate charts $\varphi_\alpha : U_\alpha \to W_\alpha$, $\varphi_\beta : U_\beta \to W_\beta$ of a submanifold M with $M \cap W_\alpha \cap W_\beta \neq \emptyset$; in that case we do indeed have exactly the above with $\varphi = \varphi_\alpha | U_{\alpha\beta}$, $\widetilde{\varphi} = \varphi_\beta | U_{\beta\alpha}$, $U = U_{\alpha\beta}$, $\widetilde{U} = U_{\beta\alpha}$, and note that in this case ψ is the transition map $\varphi_{\alpha\beta} = \varphi_\beta^{-1} \circ \varphi_\alpha : U_{\alpha\beta} \to U_{\beta\alpha}$.
- **Definition of integration of a k-form over a k-dimensional parametrized surface:**

$$(\ddagger) \qquad \int_{\varphi(\Omega)} \omega = \int_\Omega \varphi^* \omega.$$

- The fact that (with $\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}^n$ as in $(**)$)

$$(\ddagger\ddagger) \qquad \int_\Omega \varphi^* \omega = \pm \int_{\widetilde{\Omega}} \widetilde{\varphi}^* \omega.$$

with $\pm$ according as the "transition map" $\psi = \widetilde{\varphi}^{-1} \circ \varphi$ (which enables us to switch between the representations φ and $\widetilde{\varphi}$ because $\varphi = \widetilde{\varphi} \circ \psi$) is orientation preserving or reversing.
- Formal definition: M is a k-dimensional C^1 submanifold-with-boundary in $\mathbb{R}^n$ if there is a family of 1:1 C^1 maps $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in \Lambda}$ with each W_α open in $\mathbb{R}^n$ and $M \subset \cup_{\alpha \in \Lambda} W_\alpha$, and where for each $\alpha \in \Lambda$ ($\Lambda$is an index set) we have $D_1\varphi_\alpha(\underline{x}), \dots, D_k\varphi_\alpha(\underline{x})$ are l.i. for each $\underline{x} \in U_\alpha$, $\varphi_\alpha(U_\alpha) = M \cap W_\alpha$, and $\varphi_\alpha^{-1} : M \cap W_\alpha \to U_\alpha$ is continuous, and for the set U_α there are 2 possibilities:
either (a) U_α is open in $\mathbb{R}^k$ or (b) $U_\alpha = V_\alpha \cap \mathbb{R}^k_+$ where V_α is open in $\mathbb{R}^k$ and $\mathbb{R}^k_+ = \{\underline{x} \in \mathbb{R}^k : x_k \geq 0\}$ and $V_\alpha \cap (\mathbb{R}^{k-1} \times \{0\}) \neq \emptyset$. In case (a) we call φ_α an "interior coordinate chart" and in case (b) we call φ_α a boundary coordinate chart. We also write $\Lambda = \Lambda_{\mathrm{int}} \cup \Lambda_{\mathrm{bdry}}$, where $\alpha \in \Lambda_{\mathrm{int}}$ if φ_α is an interior coordinate chart, and $\alpha \in \Lambda_{\mathrm{bdry}}$ if φ_α is a boundary coordinate chart.
- Definition: ∂M =boundary of $M = \cup_{\alpha \in \Lambda_{\mathrm{bdry}}} \varphi_\alpha(U_\alpha \cap (\mathbb{R}^{k-1} \times \{0\})$ and the fact that $\partial M \neq \emptyset \Rightarrow \partial M$ is a $(k-1)$-dimensional C^1 submanifold without boundary.
- Lemma: ∂M is either empty (called manifold-without-boundary) or is a $(k-1)$-dimensional manifold-with-boundary with atlas $\{\widetilde{\varphi}_\alpha : \widetilde{U}_\alpha \to \mathbb{R}^n\}_{\alpha \in \Lambda_{\mathrm{bdry}}}$ where $\widetilde{U}_\alpha = \{\underline{x} \in \mathbb{R}^{k-1} : [\underline{x}, 0] \in U_\alpha \cap (\mathbb{R}^{k-1} \times \{0\})\}$ and $\widetilde{\varphi}_\alpha = \varphi_\alpha[\underline{x}, 0], x \in \widetilde{U}_\alpha$. $\{\widetilde{\varphi}_\alpha : \widetilde{U}_\alpha \to \mathbb{R}^n\}$ orients ∂M if $\{\varphi_\alpha : U_\alpha\}_{\alpha \in \Lambda}$ orients M.
- Lemma: $M \setminus \partial M$ is relatively open in M; that is, there is an open set $W \subset \mathbb{R}^n$ such that $M \setminus \partial M = M \cap W$. (Q.6 of hw8.)
- As in the case of C^1 manifolds without boundary we again have C^1 transition maps $\varphi_{\alpha\beta} = \beta^{-1} \circ \varphi_\alpha : U_{\alpha\beta} \to U_{\beta\alpha}$.
- For $q \in M$, the tangent space of M at q is defined by $T_q M = \{\gamma'(0) : \gamma : [0, \delta) \to \mathbb{R}^n$ is C^1 for some $\delta > 0$, and $\gamma[0, \delta) \subset M$ with $\gamma(0) = q\}$.

- Lemma: If $q \in M \setminus \partial M$, $T_q M$ is a k-dimensional subspace of $\mathbb{R}^n$ and in fact is given explicitly as span $\{D_1\varphi_\alpha(\underline{p}), \ldots, D_k\varphi_\alpha(\underline{p})\}$ for any coordinate chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ with $q \in \varphi_\alpha(U_\alpha)$ and any $\underline{p} \in U_\alpha$ with $\varphi_\alpha(\underline{p}) = q$. If $q \in \partial M$, $T_q M$ is a k-dimensional half-space of $\mathbb{R}^n$ and given explicitly as $\{\sum_{j=1}^k c_j D_j \varphi_\alpha(\underline{p}) : c_1, \ldots, c_{k-1} \in \mathbb{R},\, c_k \geq 0\}$ for any coordinate chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ with $\alpha \in \Lambda_{\text{bdry}}$ and any $\underline{p} \in U_\alpha$ with $\varphi_\alpha(\underline{p}) = q$ for some $\underline{p} \in U_\alpha \cap \mathbb{R}^{k-1} \times \{0\}$.
- Definition: An atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha\in\Lambda}$ is an orienting atlas for M ("provides an orientation for M") if $\det D\varphi_{\alpha\beta} > 0 \,\forall\, \alpha, \beta \in \Lambda$ such $\varphi_{\alpha\beta}$ is defined (i.e. for all $\alpha, \beta \in \Lambda$ such that $M \cap W_\alpha \cap W_\beta \neq \emptyset$).
- The fact that $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha\in\Lambda}$ is an orienting atlas for M and $\partial M \neq \emptyset \Rightarrow \widetilde{\mathcal{A}} = \{\widetilde{\varphi}_\alpha : \widetilde{U}_\alpha \to W_\alpha\}_{\alpha\in\Lambda_{\text{bdry}}}$ is an orienting atlas for ∂M, where $\widetilde{U}_\alpha = \{\underline{x} : [\underline{x}, 0] \in U_\alpha \cap (R^{k-1} \times \{0\})\}$ and $\widetilde{\varphi}_\alpha(\underline{x}) = \varphi_\alpha[\underline{x}, 0]$ ($[\underline{x}, 0] = (x_1, \ldots, x_{k-1}, 0)^{\mathrm{T}}$).
- Partition of Unity: Given a compact C^1 k-dimensional manifold M in $\mathbb{R}^n$ with atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha\in\Lambda}$ we can select finitely many non-negative C^∞ functions $h_1, \ldots, h_N : \mathbb{R}^n \to \mathbb{R}$ such that $\sum_{i=1}^N h_i \equiv 1$ on some open $W \supset M$ and for each $i = 1, \ldots, N$ there is $\alpha_i \in \Lambda$ with $\{\underline{x} : h_i(\underline{x}) \neq 0\}$ contained in a compact subset $K_i \subset W_{\alpha_i}$.
- Integration of functions on a compact k-dimensional submanifold M with boundary (not necessarily oriented) with atlas $\mathcal{A}$: With a partition of unity $h_1, \ldots, h_N$ as above, and motivated by $(*), (**)$ we define
$$\int_M f = \sum_{i=1}^N \int_{U_{\alpha_i}} h_i \circ \varphi_{\alpha_i} f \circ \varphi_{\alpha_i} \sqrt{\det((D\varphi_{\alpha_i})^{\mathrm{T}} D\varphi_{\alpha_i})}.$$
- Lemma: This is independent of the particular choice of the choice of partition of unity.
- The k-volume of M (k-vol(M), M as above) is defined to be $\int_M 1$ (i.e. $\int_M f$ in the special case when $f \equiv 1$).
- Integration of k-forms on a compact oriented k-dimensional manifold M oriented by the atlas $\mathcal{A}$: With a partition of unity $h_1, \ldots, h_N$ as above, and motivated by (‡), (‡‡) we define
$$\int_M \omega = \sum_{i=1}^N \int_{U_{\alpha_i}} \varphi_{\alpha_i}^*(h_i\, \omega).$$
- Lemma: This is independent of the particular choice of the choice of partition of unity.
- Stokes Thm: M is a compact oriented k-dim C^2 manifold-with-boundary in $\mathbb{R}^n$ and ∂M is oriented by $\begin{cases} \{\widetilde{\varphi}_\alpha\}_{\alpha\in\Lambda_{\text{bdry}}} & \text{if k is even} \\ \{\widehat{\varphi}_\alpha\}_{\alpha\in\Lambda_{\text{bdry}}} & \text{if k is odd}, \widehat{\varphi}_\alpha(x_1, \cdots, x_{k-1}) = \widetilde{\varphi}_\alpha(-x_1, x_2, \cdots, x_{k-1}). \end{cases}$ Then $\int_M d\omega = \int_{\partial M} \omega$ for any C^1 $(k-1)$-form ω defined on an open $V \supset M$.
- Volume form $\upsilon = \sum_{I\in\mathcal{I}_{k,n}} b_I\, dx_I$ is the k-form on M (assuming M oriented) which is defined on $M \cap W_\alpha$ by
$$b_I = \frac{(\det D\varphi_{\alpha I}) \circ \varphi_\alpha^{-1}}{\sqrt{(\det((D\varphi_\alpha)^{\mathrm{T}} D\varphi_\alpha)) \circ \varphi_\alpha^{-1}}}, \quad I \in \mathcal{I}_{k,n},$$
and this expression is independent of α (i.e. we get the same result on $M \cap W_\alpha \cap W_\beta$ if we use the chart φ_β in the definition instead of φ_α).
- Proof that υ has "length 1" in the sense that $\sum_{I\in\mathcal{I}_{k,n}} b_I^2 \equiv 1$ on M.
- (Connection between integration of functions and integration of forms.) Let M be a compact oriented k-dimensional C^1 submanifold with boundary (possibly with $\partial M = \emptyset$) and υ the volume form of M. Then for any C^0 k-form ω on M, $\int_M \omega = \int_M \langle\omega, \upsilon\rangle$, where $\langle\omega, \upsilon\rangle$ is the

inner product defined to be $\sum_{I\in\mathcal{I}_{k,n}} a_I b_I$ in case $\omega = \sum_{I\in\mathcal{I}_{k,n}} a_I\,dx_I$ and $\upsilon = \sum_{I\in\mathcal{I}_{k,n}} b_I\,dx_I$. In particular (since $\langle\upsilon,\upsilon\rangle = \sum_{I\in\mathcal{I}_{k,n}} b_I^2 \equiv 1$) we have $\int_M \upsilon = \int_M \langle\upsilon,\upsilon\rangle = \int_M 1 = k$-vol$(M)$ (which explains why υ is called the volume form).

- When $k = n-1$: M is compact oriented with volume form $\upsilon = \sum_{j=1}^n b_j\,dx_1\wedge\cdots dx_{j-1}\wedge dx_{j+1}\wedge\cdots dx_n \Rightarrow \nu = \sum_{j=1}^n (-1)^{j-1} b_j \underset{\sim}{e}_j$ is a continuous unit normal for M.
- When $k = n$, $D\varphi_\alpha$ is an $n\times n$ matrix, so $\det D\varphi_\alpha$ makes sense and is non-zero, so WLOG we can assume it is positive in U_α (otherwise replace it by $\varphi_\alpha\circ R$ where R is the reflection $(x_1,\ldots,x_n)\mapsto(-x_1,x_2,\ldots,x_n)$), in which case the atlas automatically orients M (since $\varphi_{\alpha\beta} = \varphi_\beta^{-1}\circ\varphi_\alpha$ then has Jacobian matrix of positive determinant). Also, in this case $k = n$, $V = M\setminus\partial M$ is open subset of $\mathbb{R}^n$ and $\partial V = \partial M$ (see hw10, Q.4). In this case Stokes theorem (applied to the $(n-1)$-form $\omega = \sum_{i=1}^n (-1)^{i-1} a_i\,dx_{\hat{i}}$, where a_j are C^1 in an open $U \supset V\cup\partial V$), implies the "divergence theorem" that
$$\int_V \operatorname{div}\underset{\sim}{a} = \int_{\partial V} \underset{\sim}{a}\cdot\nu,$$
where $\underset{\sim}{a} = (a_1,\ldots,a_n)^{\mathrm{T}}$, $\operatorname{div}\underset{\sim}{a} = \sum_{j=1}^n D_j a_j$ and ν is the unit normal of $\partial V(=\partial M)$ pointing out of V.
- When $k = 2, n = 3$, Stokes Theorem (applied to the 1-form $\sum_{j=1}^n a_j\,dx_j$) gives
$$\int_M (\nabla\times\underset{\sim}{a})\cdot\nu = \int_{\partial M} \tau\cdot\underset{\sim}{a}\,ds,$$
where s is the arc-length parameter on ∂M, $\nabla\times\underset{\sim}{a} = (D_2a_3 - D_3a_2, D_3a_1 - D_1a_3, D_1a_2 - D_2a_1)^{\mathrm{T}}$, ν is the unit normal of M, and τ is the unit tangent of ∂M, oriented so that $\nu = \tau\times\gamma$, where at each point $q\in\partial M$, $\gamma|_q$ is the unit vector in T_qM which is normal to τ and points into M. (Q.2 of hw10.)
- Applications of Stokes Theorem, including proofs of the fundamental theorem of algebra and the Brouwer fixed point theorem.