ME111
Instructor: Peter Pinsky
Lecture #18
December 1, 1999

Today's Topic

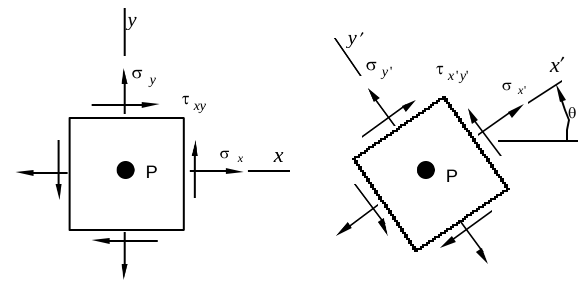
- Course Review (starting with stress transformations from Lect. 5)

Note: Today's lecture is #18 but is class #20, since we had two guest lectures.

Main topics for the final exam:

1. Ductile failure (plastic yielding) criterion
2. Brittle failure criterion
3. Stress concentrations
4. Fracture analysis (LEFM and effects of local yielding)
5. Uniaxial and multiaxial stress-life high cycle fatigue (HCF) analysis
6. Bucking and failure of columns.

Exam -- open notes, open text book (make sure you have a copy or sit by someone who doesn't mind sharing theirs). There wil be (most likely) 5 questions, each of which will be relatively short -- don't allow yourself to get caught up spending too much time on one question -- move on! Good luck (not that you need it).



5.4 Transformation of Stress in 2-d

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

5.5 Principal Stresses in 2-d

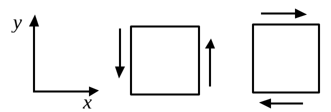
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

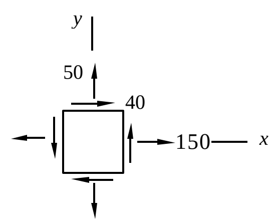
5.5 Maximum Shear Stresses

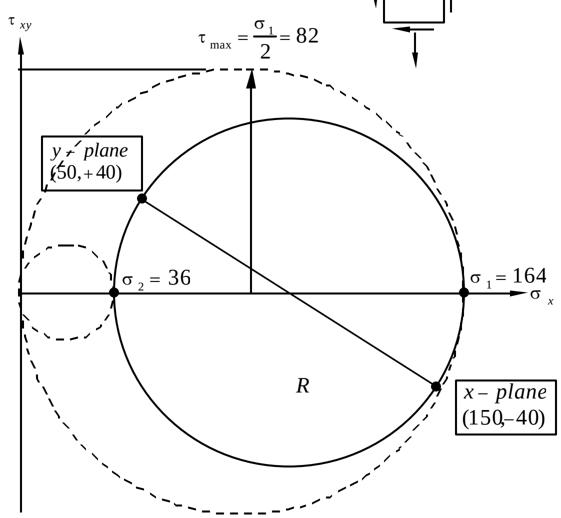
$$\tau_{max,min} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

6.1 Mohr's Circle for Plane Stress

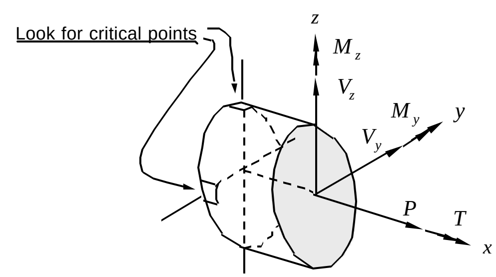


Stress tensor:	positive	positive
Mohr's circle:	negative	positive





7.1 Combined Stresses



Normal stress due to axial force	$\sigma = \frac{P}{A}$		
Normal stress due to bending moment	$\sigma = \frac{My}{I}$	$I = \frac{\pi D^4}{64}$	$\sigma_{max} = \frac{Mc}{I}$
Shear stress due to bending moment	$\tau = \frac{VQ}{It}$	$Q = \overline{y}'A'$	$\tau_{max} = \frac{4V}{3A}$
Shear stress due to torque	$\tau = \frac{Tr}{J}$	$J = \frac{\pi D^4}{32}$	$\tau_{max} = \frac{Tc}{J}$

8.1 Triaxial Stress States

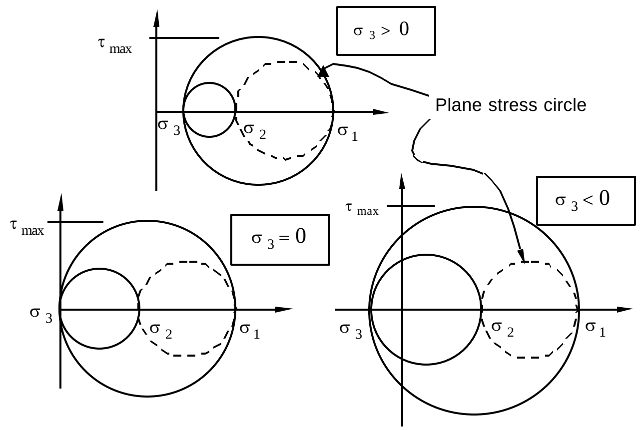
• The maximum shear stresses are:

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2}$$

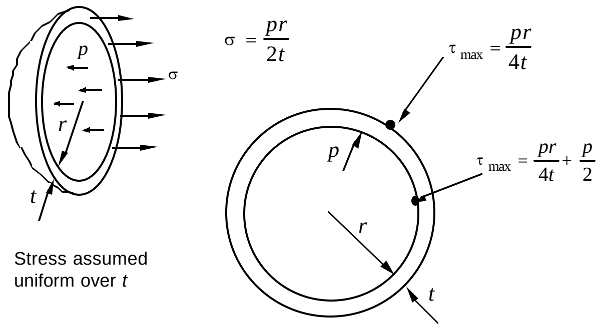
$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2} \qquad \tau_{max} = \max(\tau_{13}, \tau_{21}, \tau_{32})$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

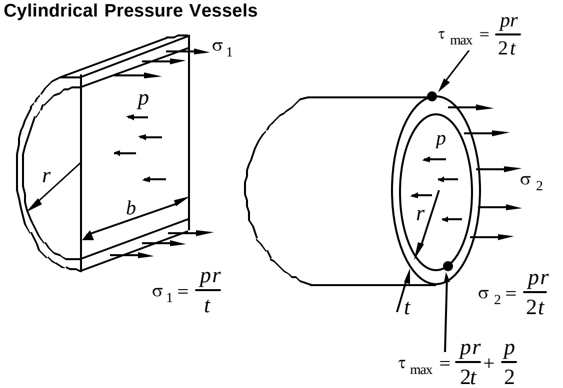
• Graphically, using Mohr's circle, we have:



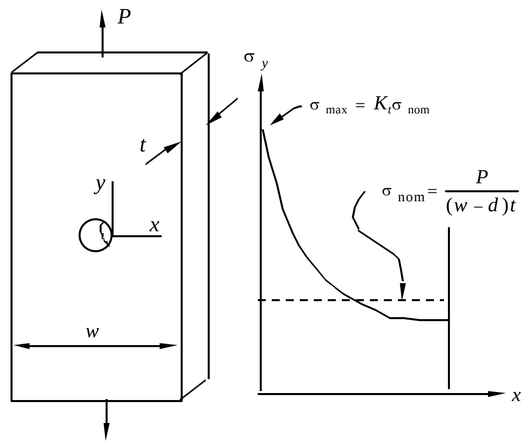
8.3 Thin-Walled Spherical Pressure Vessels



8.4 Cylindrical Pressure Vessels



8.5 Stress Concentration Effects



• Define the nominal stress to be:

$$\sigma_{nom} = \frac{P}{A_{nom}} = \frac{P}{(w-d)t}$$

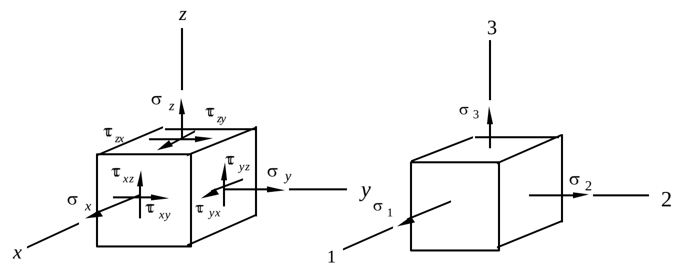
• Maximum stress near hole is:

$$\sigma_{max} = K_t \sigma_{nom}$$

where K_t is called the stress concentration factor.

9.2 Three-Dimensional Stress States

• It's useful to think about 3-d stress states using principal stresses:



Principal stresses

$$\boldsymbol{\sigma}^3 - I_1\boldsymbol{\sigma}^2 + I_2\boldsymbol{\sigma} - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

Maximum shear stresses

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2}$$

$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2} \qquad \tau_{max} = \max(\tau_{13}, \tau_{21}, \tau_{32})$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

9.3 von Mises (distortion energy) yield criterion

$$S_y = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1}$$

$$S_y = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}}$$

For plane stress $\sigma_2 = 0$

$$S_y = \sqrt{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2}$$

$$S_y = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2}$$

Von Mises Effective Stress

$$\sigma' = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1}$$

$$\sigma' = \sqrt{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2}$$

Factor of Safety Against Yielding

$$\frac{S_y}{N} = \sigma'$$

10.1 Maximum Shear Stress Criterion (Tresca Condition)

$$\max\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2}\right) = \frac{S_y}{2}$$

or

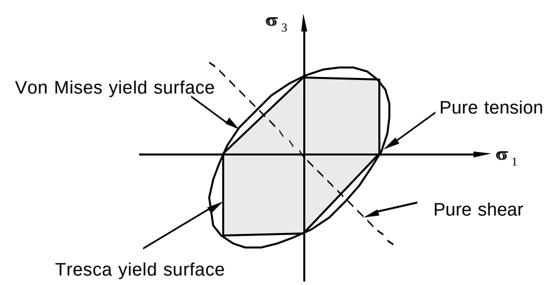
$$\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = S_y$$

For plane stress with $\sigma_2 = 0$

$$\max(|\sigma_1|, |\sigma_3|, |\sigma_3 - \sigma_1|) = S_y$$

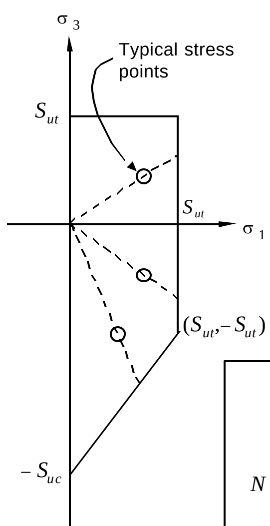
Factor of Safety Against Yielding

$$\frac{S_y}{N} = \max(|\sigma_1|, |\sigma_3|, |\sigma_3 - \sigma_1|)$$



10.2 Brittle Failure Under Static Loads

Modified Mohr theory:



First quadrant

$\sigma_1 \geq 0, \quad \sigma_3 \geq 0$

$$N = \frac{S_{ut}}{\sigma_1}$$

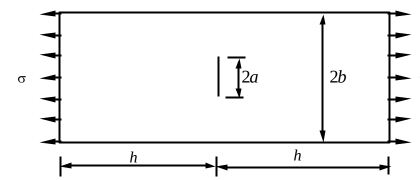
Fourth quadrant

$\sigma_1 \geq 0, \quad \sigma_3 < 0$

$$N = \frac{S_{ut}}{\sigma_1} \qquad |\sigma_3| \leq \sigma_1$$

$$N = \frac{S_{ut} S_{uc}}{S_{uc}\sigma_1 - S_{ut}(\sigma_1 + \sigma_3)} \qquad |\sigma_3| > \sigma_1$$

11.3 State of Stress Near a Crack Tip



Stress intensity factor

$$K = \beta\sigma\sqrt{\pi a}$$

11.4 Fracture Toughness

We have failure by fracture when K grows to $K = K_C$

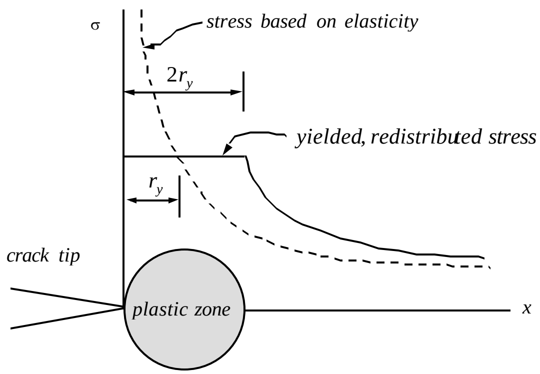
$K < K_C$ Crack will not propagate

$K \geq K_C$ Crack will propagate

11.5 Factor of Safety in Fracture

$$N = \frac{K_C}{K}$$

12.3 Effects of Small Scale Yielding at the Crack Tip

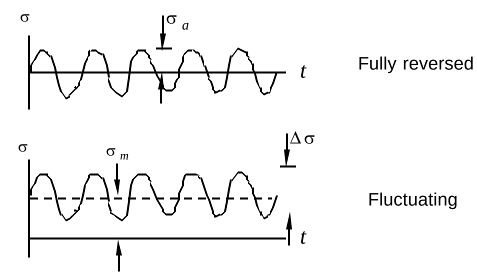


$$a_{eff} = a + 2r_y = a + \frac{1}{\pi}\left(\frac{K}{S_y}\right)^2$$

$$K = \beta\sigma\sqrt{\pi a_{eff}}$$

15.1 Overview of Approach to HCF Analysis

I Fully reversed Uniaxial stress	II Fluctuating Uniaxial stress
III Fully reversed Multiaxial stresses	IV Fluctuating Multiaxial stresses



Alternating stress $\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}$

Mean stress $\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$

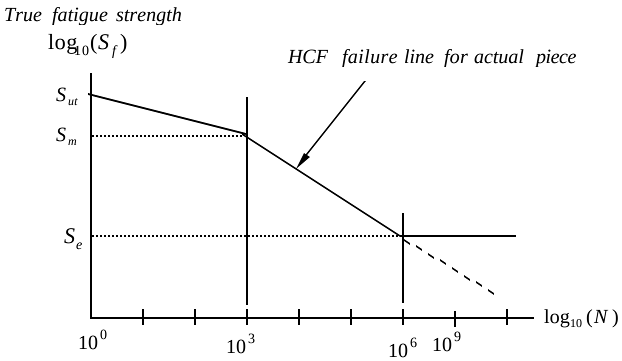
15.2 Fully Reversed Uniaxial Stress HCF

Corrected endurance limit:

$$S_e = C_{load}C_{size}C_{surf}C_{temp}C_{reliab}S_e'$$

Corrected fatigue strength:

$$S_f = C_{load}C_{size}C_{surf}C_{temp}C_{reliab}S_f'$$



Fatigue strength at N>10^3 cycles:

$$S_f = aN^b$$
$$\log_{10}(a) = \log_{10}(S_m) - 3b$$
$$b = \frac{1}{3-\log_{10}(N_2)}\log\left(\frac{S_m}{S_e}\right)$$

Factor of Safety

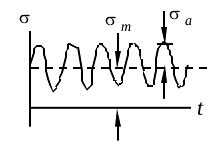
- Design for HCF:

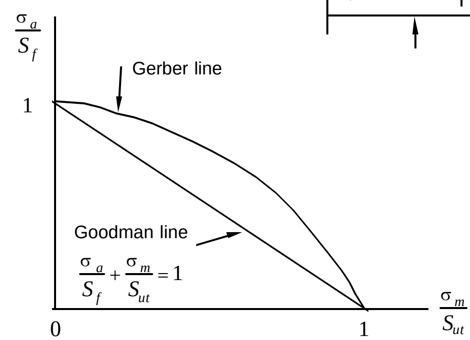
$$N_f = \frac{S_f}{\sigma_a}$$

- Design for Infinite Life Fatigue:

$$N_f = \frac{S_e}{\sigma_a}$$

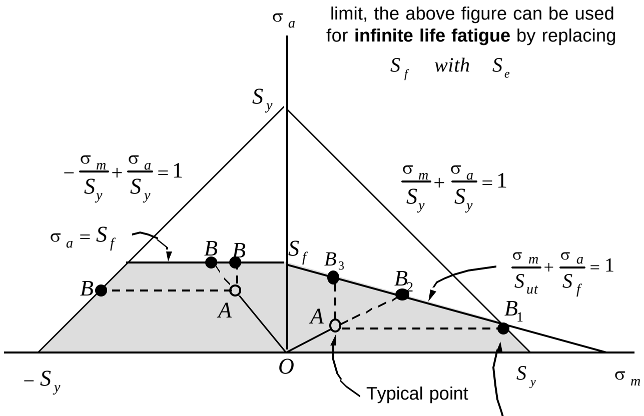
15.3 Fluctuating Uniaxial Stress HCF





Factor of Safety

Important note: If the material exhibits an endurance limit, the above figure can be used for **infinite life fatigue** by replacing S_f with S_e



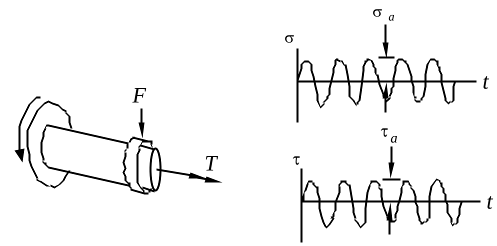
Example

The alternating stress and mean stress can increase in a fixed ratio:

$$\frac{\sigma_m}{S_y}+\frac{\sigma_a}{S_y}=1 \Rightarrow \sigma_{m,\,fail}=\left(1-\frac{\sigma_a}{S_y}\right)S_y$$

$$N_f=\frac{\sigma_{m,fail}}{\sigma_m}=\frac{\left(1-\frac{\sigma_a}{S_y}\right)S_y}{\sigma_m}$$

15.4 Fully Reversed Multiaxial Stresses HCF



Convert stresses to von Mises (effective) alternating stress:

$$\sigma'_a=\sqrt{\frac{(\sigma_{ax}-\sigma_{ay})^2+(\sigma_{ay}-\sigma_{az})^2+(\sigma_{az}-\sigma_{ax})^2+6(\tau_{axy}^2+\tau_{ayz}^2+\tau_{azx}^2)}{2}}$$

$$\sigma'_a=\sqrt{\sigma_{ax}^2+\sigma_{ay}^2-\sigma_{ax}\sigma_{ay}+3\tau_{axy}^2}$$

Factor of Safety

- Design for HCF:

$$N_f=\frac{S_f}{\sigma'_a}$$

$$S_f=aN^b$$

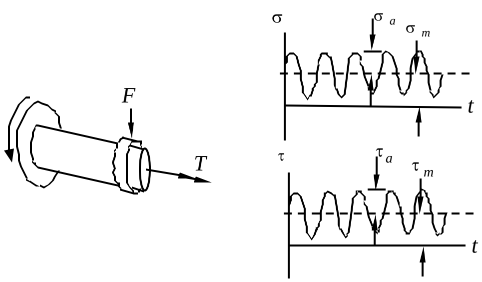
- Design for Infinite Life Fatigue:

$$N_f=\frac{S_e}{\sigma'_a}$$

$$\log_{10}(a)=\log_{10}(S_m)-3b$$

$$b=\frac{1}{3-\log_{10}(N_2)}\log\left(\frac{S_m}{S_e}\right)$$

15.5 Fluctuating Multiaxial Stresses HCF



synchronous (in phase) von Mises (effective) alternating stress in plane stress:

$$\sigma'_a = \sqrt{\sigma_{ax}^2 + \sigma_{ay}^2 - \sigma_{ax}\sigma_{ay} + 3\tau_{axy}^2}$$

Effective mean stress in plane stress:

$$\sigma'_m = \sigma_{mx} + \sigma_{my} + \sigma_{mz}$$ (Sines method)

$$\sigma'_m = \sqrt{\sigma_{mx}^2 + \sigma_{my}^2 - \sigma_{mx}\sigma_{my} + 3\tau_{mxy}^2}$$ (von Mises method)

Factor of Safety

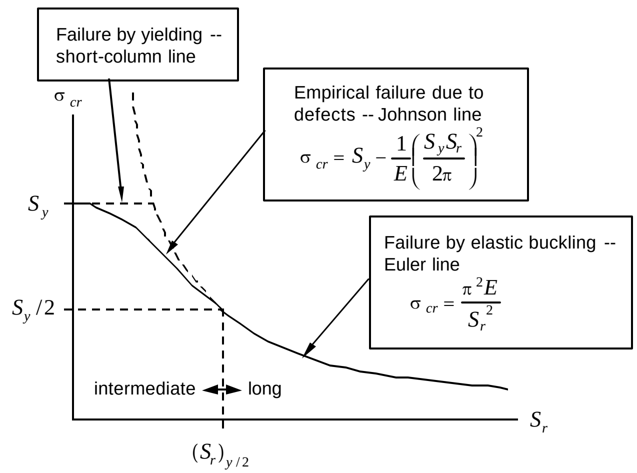
- Design for HCF:

  Same as Fluctuating Uniaxial Stress HCF

- Design for Infinite Life Fatigue:

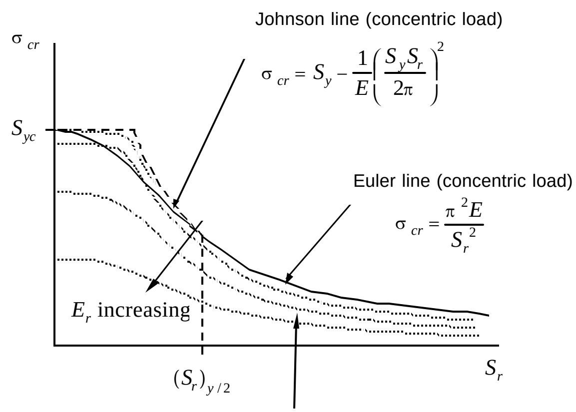
  Same as Fluctuating Uniaxial Stress HCF

17.1-5 Euler and Johnson Formula



Intermediate column	**Long column**
$S_r \le \pi\sqrt{\frac{2E}{S_y}}$	$S_r > \pi\sqrt{\frac{2E}{S_y}}$
$\sigma_{cr} = S_y - \frac{1}{E}\left(\frac{S_y S_r}{2\pi}\right)^2$	$\sigma_{cr} = \frac{\pi^2 E}{S_r^2}$

17.7 Secant Column Formula



$$\sigma_{cr} = S_y - \frac{1}{E}\left(\frac{S_y S_r}{2\pi}\right)^2$$

$$\sigma_{cr} = \frac{\pi^2 E}{S_r^2}$$

Secant line (eccentric load)

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{S_{yc}}{1 + E_r \sec\left(\frac{L_{eff}}{2\rho}\sqrt{\frac{P_{cr}}{EA}}\right)}$$