ME111
Instructor: Peter Pinsky
Class #21
November 13, 2000

Today's Topics

- Failure of ductile materials under static loading.
- The von Mises yield criterion.
- The maximum shear stress criterion.

Reading Assignment

Juvinall, 6.5 – 6.8

Problem Set #7. Due Wednesday, November 22.

1. The stress (in kpsi) at a point is given by $\sigma_1 = 10, \sigma_2 = 0, \sigma_3 = -20$
   Calculate the factor of safety against failure if the material is:
   (a) Brittle with S_{ut} = 50, S_{uc} = 90 and using modified-Mohr theory.
   (b) Ductile with S_y = 40 using (i) the max. shear stress criterion, and (ii) von Mises criterion.

2. Consider two designs of a lug wrench for an automobile: (a) single ended, (b) double ended. The distance between points A and B is 12 in. and the handle diameter is 0.625 in. What is the maximum force possible before yielding the handle if S_y = 45 kpsi?

3. A storage rack is to be designed to hold a roll of industrial paper. The weight of the roll is 53.9 kN, and the length of the mandrel is 1.615 m. Determine suitable dimensions for *a* and *b* to provide a factor of safety of 1.5 if:
   (a) The beam is a ductile material with S_y = 300 Mpa
   (b) The beam is a brittle material with S_{ut} = 150 Mpa, S_{uc} = 570 Mpa.

4. For the problem in Example 19.1 (torsion-bar spring), what diameter d will provide a factor of safety of N = 3 against yielding based on von Mises with S_y = 150 MPa.

5. Repeat Example 21.2 (a) and (b) using the maximum shear stress criterion.

6. For the beam shown, determine the factor of safety for:
   (a) Ductile material with S_y = 300 Mpa
   (b) Brittle material with S_{ut} = 150 Mpa, S_{uc} = 570 Mpa.

Problem 2

Problem 3

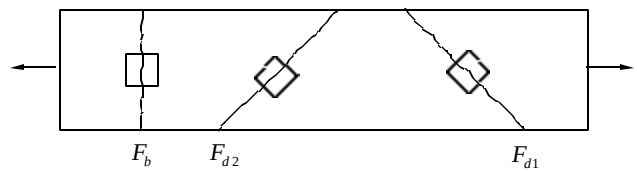
Problem 6

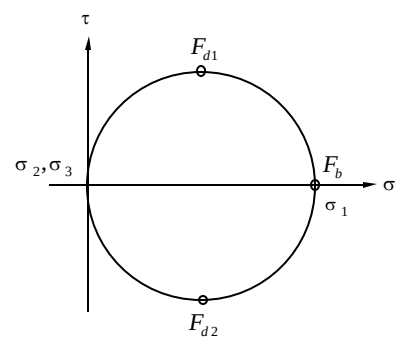
21.1 Failure of Ductile Materials Under Static Loading

Static Loads:

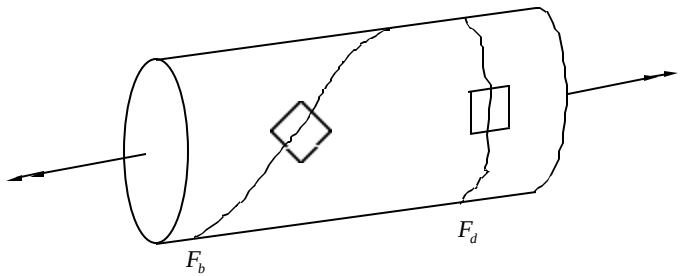
- Brittle materials fail by **cracking** or **crushing** and are typically limited by their **tensile** strengths
- Ductile materials fail by **yielding** and are typically limited by their **shear** strengths

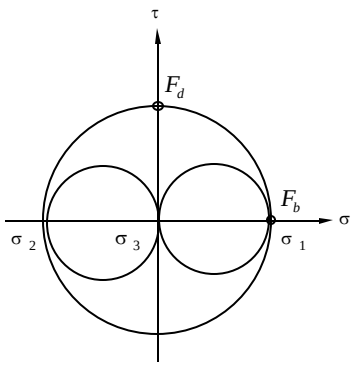
Recall Uniaxial Tension Test:





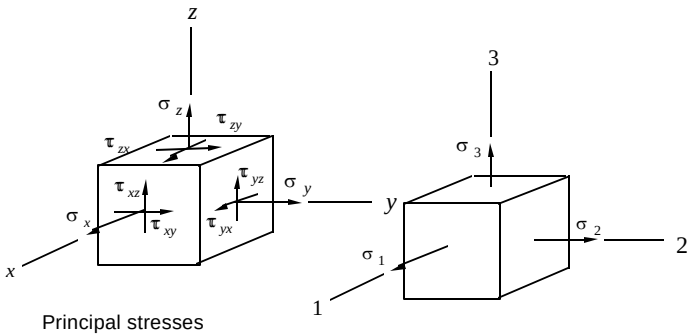
Recall Torsion Test:





21.2 Three-Dimensional Stress States

- It's useful to think about 3-d stress states using **principal stresses**:



Principal stresses

$$\boldsymbol{\sigma}^3 - I_1\boldsymbol{\sigma}^2 + I_2\boldsymbol{\sigma} - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

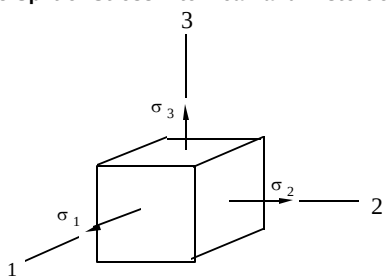
Maximum shear stresses

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2}$$

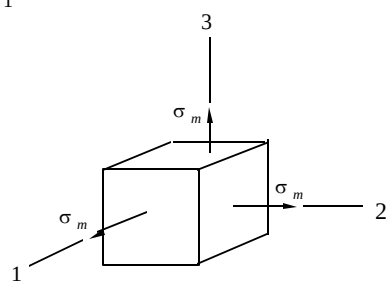
$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2} \qquad \tau_{max} = \max(\tau_{13}, \tau_{21}, \tau_{32})$$

$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

21.3 Split of Stress into Mean and Distortional Components

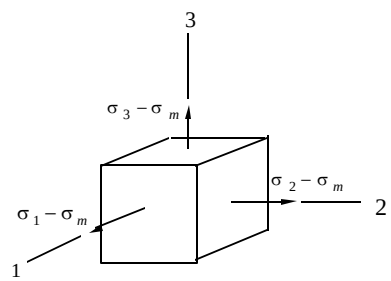


Given principal stress state



$$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Mean stress produces volumetric strain



Produces distortional strain responsible for plastic yielding

21.4 Yielding of Ductile Materials

- A ductile material yields when a **yield criterion** is exceeded.
- Two yield criteria are important:

The von Mises Yield Theory

- The strain energy per unit volume of an elastic body has the form:

$$U = U_m + U_d$$

$$U_m = \frac{1-2\nu}{6E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$U_d = \frac{1+\nu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right]$$

where U_h is the strain energy per unit volume associated with pure <u>volume change</u> (dilation) due to the mean (hydrostatic) stress:

$$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

and where U_d is the strain energy per unit volume associated with <u>pure distortion</u>

- Note that in a <u>tensile test at yield</u>:

$$\sigma_1 = S_y, \quad \sigma_2 = \sigma_3 = 0$$

and, in this special case,

$$U_d = \frac{1+\nu}{3E} S_y^2$$

- The von Mises yield criterion predicts failure in a general 3-d stress state when the distortion energy per unit volume U_d is equal to the distortion energy per unit volume in the tensile test specimen at failure, i.e.

$$\boxed{U_d = \frac{1+\nu}{3E} S_y^2}$$

$$\frac{1+\nu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right] = \frac{1+\nu}{3E} S_y^2$$

or,

$$S_y = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1}$$

In terms of x, y and z stresses

$$S_y = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}}$$

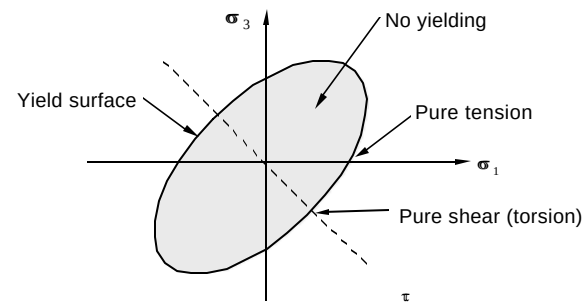
For plane stress, $\sigma_2 = 0$ Note, this is rather arbitrary but we'll work with this as the plane stress state.

$$\boxed{S_y = \sqrt{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2}}$$

In terms of x, y and z stresses

$$\boxed{S_y = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2}}$$

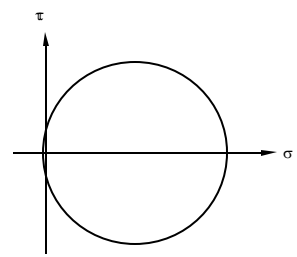
Von Mises ellipse (plane stress)



Special cases:

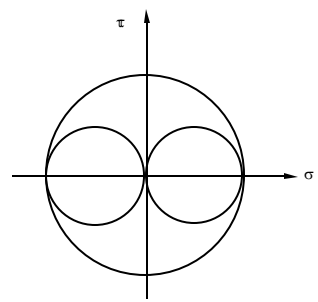
Pure tension

$$\boxed{\begin{aligned} &\sigma_2 = \sigma_3 = 0 \\ &S_y = \sigma_1 \end{aligned}}$$



Pure shear (torsion)

$$\boxed{\begin{aligned} &\sigma_1 = -\sigma_3 = \tau_{max}, \quad \sigma_2 = 0 \\ &S_y = \sqrt{3}\sigma_1 = \sqrt{3}\tau_{max} \end{aligned}}$$



Von Mises Effective Stress

Convenient to introduce the **von mises effective stress** σ'

$$\sigma' = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1}$$

(The von Mises effective stress σ' is defined as the uniaxial tensile stress that would create the same distortion energy U_d as is created by the the actual combination of applied stress)

von Mises (distortion energy) Yield Criterion

The von Mises yield criterion predicts failure when:

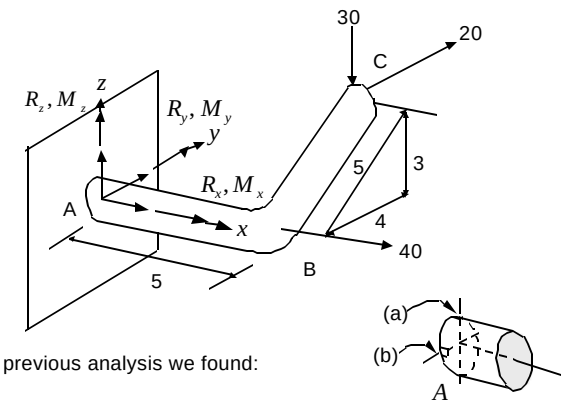
$$\boxed{S_y = \sigma'}$$

Factor of Safety Against Yielding

$$\boxed{\frac{S_y}{N} = \sigma'}$$

Example 21.1

For the bracket shown, determine the von Mises stress and factor of safety against yielding at points (a) and (b) if S_y=10,000.



From previous analysis we found:

At (a): $\sigma_1 = 271, \sigma_3 = -7590$

$$\sigma' = \sqrt{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2} = 7{,}729$$

$$N = \frac{S_y}{\sigma'} = \frac{10{,}000}{7{,}729} = 1.3$$

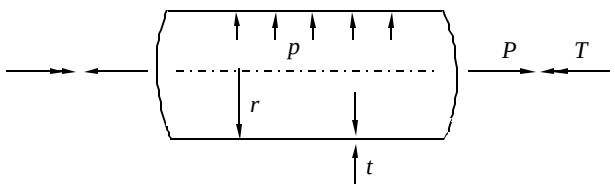
At (b): $\sigma_1 = 5096, \sigma_3 = -343$

$$\sigma' = \sqrt{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2} = 5{,}274$$

$$N = \frac{S_y}{\sigma'} = \frac{10{,}000}{5{,}274} = 1.9$$

Example 21.2

A thin-walled cylindrical pressure vessel is subject to an axial force and torque loads as shown:



(a) Given:

$$p = 15\,MPa,\ r = 35mm,\ t = 3mm,$$
$$T = 450\,kN \cdot mm,\ S_y = 290\,MPa$$

Determine the range of values of the axial load *P* which will provide a factor of safety against yielding of at least 1.4 based on the von Mises criterion.

(b) Given:

$$p = 20\,MPa,\ t = 5mm,$$
$$T = 800kN \cdot mm,\ P = 100\,kN,\ S_y = 290\,MPa$$

Determine the range of values of the radius *r* which will provide a factor of safety against yielding of at least 1.4 based on the von Mises criterion.

21.5 Maximum Shear Stress Criterion (Tresca Condition)

- Another important criteria is based on the theory that shear stress controls yielding (in contrast to von Mises theory based on distortion energy).
- This theory was developed before the von Mises criterion and in practice is slightly more conservative.
- The theory is easy to use in an "analytical" setting but is not well suited to use in finite element codes because of the corners in the yield surface (see below).
- The maximum shear stress theory states that yielding will occur when the maximum shear stress reaches the shear stress in a uniaxial test specimen at yield, i.e.

$$\max\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2}\right) = \frac{S_y}{2}$$

or

$$\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = S_y$$

For plane stress with $\sigma_2 = 0$

$$\max(|\sigma_1|, |\sigma_3|, |\sigma_3 - \sigma_1|) = S_y$$

Factor of Safety Against Yielding

$$\boxed{\frac{S_y}{N} = \max(|\sigma_1|, |\sigma_3|, |\sigma_3 - \sigma_1|)}$$

Remark: von Mises is the preferred theory

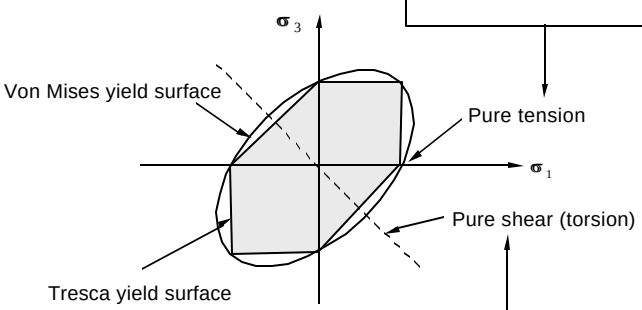
Von Mises and Tresca for Plane Stress

Stress states inside the yield surface have not yielded

$\sigma_3 = 0$

von Mises: $\sigma_1 = S_y$

Tresca: $\sigma_1 = S_y$



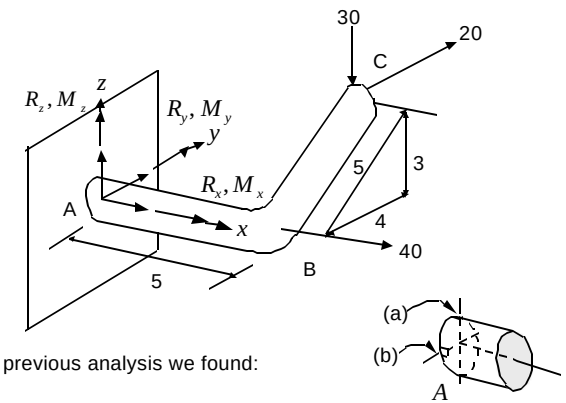
$\sigma_1 = -\sigma_3$

von Mises: $\sigma_1 = \frac{1}{\sqrt{3}} S_y$

Tresca: $\sigma_1 = \frac{1}{2} S_y$

Example 21.3

For the bracket shown, determine the factor of safety against yielding using the maximum shear stress theory at points (a) and (b) if S_y=10,000.



From previous analysis we found:

At (a): $\boldsymbol{\sigma}_1 = 271, \boldsymbol{\sigma}_3 = -7590$

$$N = \frac{S_y}{\max(|\sigma_1|, |\sigma_3|, |\sigma_3 - \sigma_1|)} = \frac{10{,}000}{7{,}861} = 1.27$$

At (b): $\boldsymbol{\sigma}_1 = 5096, \boldsymbol{\sigma}_3 = -343$

$$N = \frac{S_y}{\max(|\sigma_1|, |\sigma_3|, |\sigma_3 - \sigma_1|)} = \frac{10{,}000}{5{,}439} = 1.84$$