ME111
Instructor: Peter Pinsky
Class #7
October 11, 2000

Today's Topics

- Plane stress
- Stress transformation; Mohr's circle
- Principal stresses; maximum shear stress.

Reading Assignment

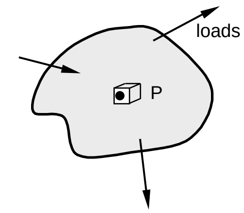
Problem Set #3 **(**Due October 18, 2000)

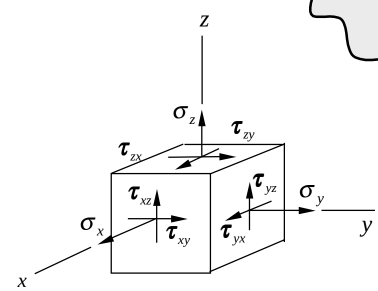
Juvinall 4.12
4.17
4.21
4.31
4.34

7.1 Stresses in Three Dimensions

- Stress at point is is expressed in terms of *components* with respect to a set of fixed coordinate axes.

- Consider the stress at a typical point P:





•From equilibrium we know

$$\tau_{xy} = \tau_{yx},$$
$$\tau_{yz} = \tau_{zy},$$
$$\tau_{zx} = \tau_{xz}$$

•There are 3 components of normal stress

$$\sigma_x, \sigma_y, \sigma_z$$

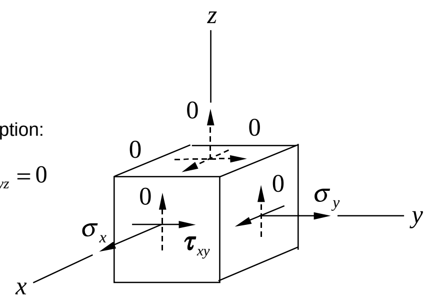
•There are 3 components of shear stress

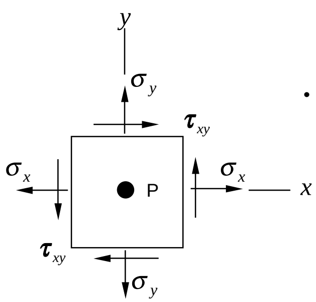
$$\tau_{xy}, \tau_{yz}, \tau_{zx}$$

7.2 Stress in Two Dimensions (Plane Stress)

• Plane stress assumption:

$\sigma_{zz} = 0,\ \tau_{zx} = 0,\ \sigma_{yz} = 0$



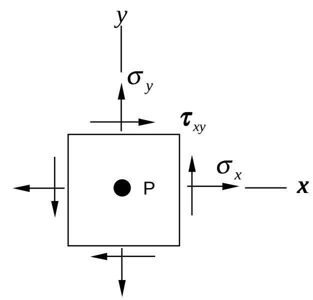


• Positive stresses in plane stress

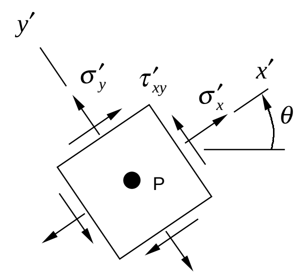
7.3 Transformation of Stress in Two Dimensions

• We need to understand how the components of the stress tensor change with rotation of coordinate axes.

Given



Find



• By considering equilibrium, we can relate the components of the stress at P referred to both coordinate systems (see E14 notes and text):

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

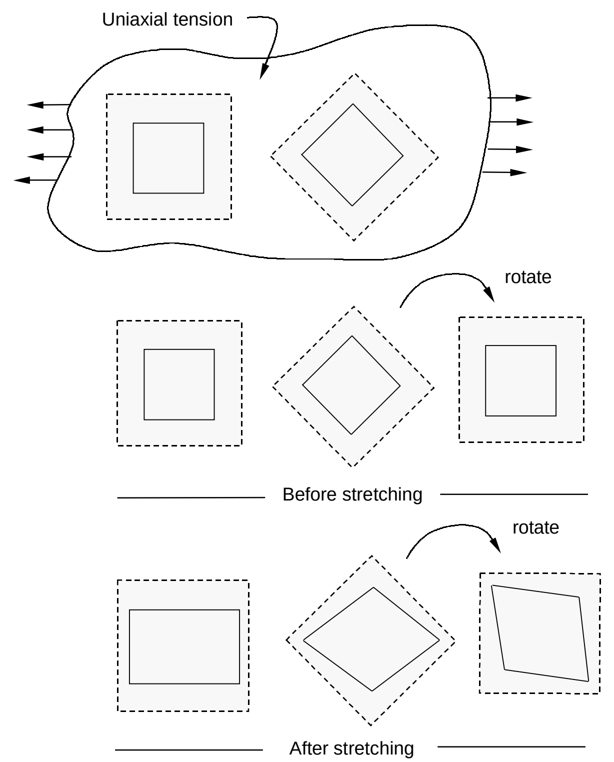
• Observe:

(1) $$\sigma_{y'} = \sigma_{x'}(\text{at } \theta = \theta + 90^0)$$
$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

(2) $\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$ -- called a *stress invariant*

Thought Experiment on Stress Transformation

(Why pure stretching also produces shearing)



Example 7.1

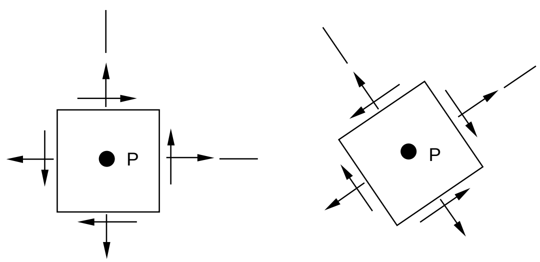
Given $\boldsymbol{\sigma}_x = 150\,MPa,\ \boldsymbol{\sigma}_y = 50\,MPa,\ \boldsymbol{\tau}_{xy} = 40\,MPa$

Determine the stresses on an element oriented at $\boldsymbol{\theta} = 40^0$

$$\sigma_{x'} = \frac{150+50}{2} + \frac{150-50}{2}\cos 80^0 + 40\sin 80^0 = 148.1$$

$$\sigma_{y'} = \frac{150+50}{2} - \frac{150-50}{2}\cos 80^0 - 40\sin 80^0 = 51.9$$

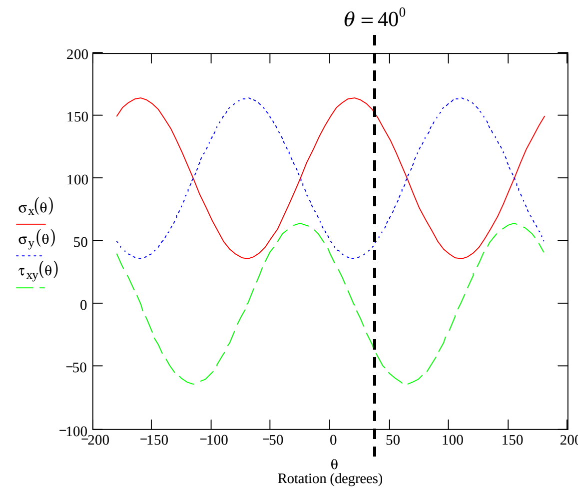
$$\tau_{x'y'} = -\frac{150-50}{2}\sin 80^0 + 40\cos 80^0 = -42.3$$



$$\sigma_{x'} = \frac{150+50}{2} + \frac{150-50}{2}\cos 2\theta + 40\sin 2\theta$$

$$\sigma_{y'} = \frac{150+50}{2} - \frac{150-50}{2}\cos 2\theta - 40\sin 2\theta$$

$$\tau_{x'y'} = -\frac{150-50}{2}\sin 2\theta + 40\cos 2\theta$$



7.4 Principal Stresses in 2-d

- There will exist values of θ for which the normal stress components are maximum (minimum):

$$\frac{d\sigma_{x'}}{d\theta} = 0 \quad \Rightarrow \quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- Provides two angles θ_{p1} and θ_{p2} which differ by 90^0 and define the *principal planes*
- Using these angles in the expression for the transformed stress gives

$$\theta_{p1} \rightarrow \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} := \sigma_1$$

$$\theta_{p2} \rightarrow \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} := \sigma_2$$

- Note that: (1) Shear stress on the principal planes is zero.
  (2) $\sigma_1 > \sigma_2$

- Question -- Given the two roots of: $\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$

how will you decide which angle defines which principal plane?

Answer -- take either root and substitute it into:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

and compare $\sigma_{x'}$ with the known values of σ_1 and σ_2

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7.5 Maximum Shear Stresses

- There will exist values of θ for which the shear stress components are maximum (minimum):

$$\frac{d\tau_{x'y'}}{d\theta} = 0 \quad \Rightarrow \quad \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

- Provides two angles θ_{s1} and θ_{s2} which differ by 90^0 and define
- Using these angles in the expression for the transformed stress gives

$$\theta_{s1} \rightarrow \tau_{x'y'} = +\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} := \tau_{max}$$

$$\theta_{s2} \rightarrow \tau_{x'y'} = -\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} := \tau_{min}$$

- Note that:

(1) $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$ (useful result!)

(2) Mean stress on **both** planes of max/min shear stress is: $\frac{\sigma_1 + \sigma_2}{2}$

(3) Observe $\sigma_x + \sigma_y = \sigma_1 + \sigma_2$

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Example 7.2

Given $\boldsymbol{\sigma}_x = 150\,MPa,\ \boldsymbol{\sigma}_y = 50\,MPa,\ \boldsymbol{\tau}_{xy} = 40\,MPa$

Determine:
(a) the principal planes and principal stresses, and
(b) the planes of maximum shear stress and the maximum shear stress value.

(a) Principal planes and stresses:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$= \frac{150+50}{2} \pm \sqrt{\left(\frac{150-50}{2}\right)^2 + 40^2}$$
$$= 164.0,\ 36.0$$

Now let's find the principal planes:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \Rightarrow \quad 2\theta_p = 38.66^0,\ 38.66^0 \pm 180^0$$
$$\Rightarrow \quad \theta_p = 19.33^0, 19.33^0 \pm 90^0$$

Consider $\theta_p = 19.33^0$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= \frac{150+50}{2} + \frac{150-50}{2}\cos 2(19.33^0) + 40\sin 2(19.33^0)$$
$$= 164.0$$
$$\Rightarrow \quad \theta_{p1} = 19.33^0, \quad \theta_{p2} = 109.33^0$$

(b) Maximum shear stresses:

$$\tau_{\max/\min} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{150-50}{2}\right)^2 + 40^2} = \pm 64.0$$

Now let's find the maximum shear planes:

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad \Rightarrow \quad 2\theta_s = -51.34^0,\ -51.34^0 \pm 180^0$$
$$\Rightarrow \quad \theta_s = -25.67^0, -25.67^0 \pm 90^0$$

Consider $\theta_s = -25.67^0$

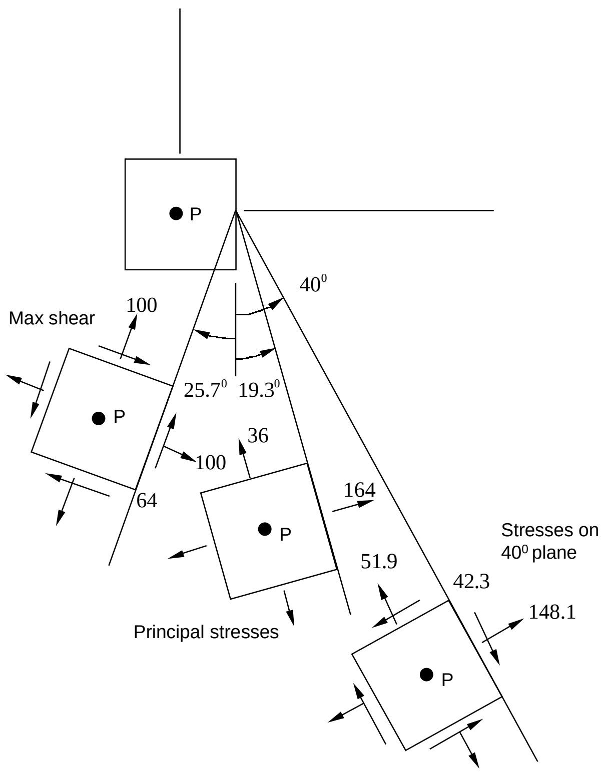
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$= -\frac{150-50}{2}\sin 2(-25.67) + 40\cos 2(-25.67) = +64.0$$
$$\Rightarrow \quad \theta_{s1} = -25.67^0, \quad \theta_{s2} = 64.33^0$$

Normal stress = $\dfrac{\sigma_1 + \sigma_2}{2} = \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{150+50}{2} = 100$

Finally, lets check that:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} \quad \rightarrow \quad +64 = \frac{164-36}{2} = \text{true}$$

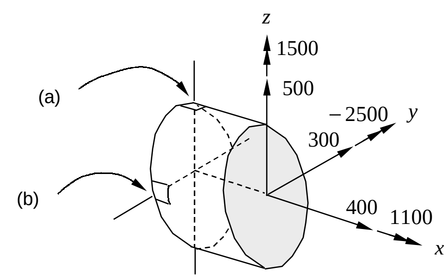
Summary



Example 7.3

Returning to Example X.X, determine the principal stresses and maximum shear stresses at points (a) and (b).

From example X.X, we have:



σ_x	−7,319	4,753
σ_y	0	0
τ_{xy}	−1,434	−1,283

Point (a)

Mathcad evaluation

$\sigma_x := -7319$

$\sigma_y := 0$

$\tau_{xy} := -1434$

$\sigma_m := \frac{(\sigma_x + \sigma_y)}{2}$

$\sigma_d := \frac{(\sigma_x - \sigma_y)}{2}$

$\rho := \sqrt{\sigma_d^2 + \tau_{xy}^2}$

$\sigma_1 := \sigma_m + \rho$

$\sigma_2 := \sigma_m - \rho$

$\theta_p := 0.5 \cdot \text{atan}\left(\frac{\tau_{xy}}{\sigma_d}\right)$

$\sigma_{check} := \sigma_m + \sigma_d \cdot \cos(2 \cdot \theta_p) + \tau_{xy} \cdot \sin(2 \cdot \theta_p)$

$\tau_{check} := -\sigma_d \cdot \sin(2 \cdot \theta_p) + \tau_{xy} \cdot \cos(2 \cdot \theta_p)$

$\sigma_1 = 270.93$

$\sigma_2 = -7589.93$

$\theta_p \cdot \frac{180}{\pi} = 10.7$

$\sigma_{check} = -7589.93$

$\tau_{check} = 0$

$\tau_1 := \rho$

$\tau_2 := -\rho$

$\theta_s := 0.5 \cdot \text{atan}\left(\frac{-\sigma_d}{\tau_{xy}}\right)$

$\tau_{check} := -\sigma_d \cdot \sin(2 \cdot \theta_s) + \tau_{xy} \cdot \cos(2 \cdot \theta_s)$

$\tau_1 = 3930.43$

$\tau_2 = -3930.43$

$\theta_s \cdot \frac{180}{\pi} = -34.3$

$\tau_{check} = -3930.43$

$\sigma_{normal} := \sigma_m$

$\sigma_{normal} = -3659.5$

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Point (b)

Mathcad evaluation

$\sigma_x := 4753$

$\sigma_y := 0$

$\tau_{xy} := -1283$

$\sigma_m := \frac{(\sigma_x + \sigma_y)}{2}$

$\sigma_d := \frac{(\sigma_x - \sigma_y)}{2}$

$\rho := \sqrt{\sigma_d^2 + \tau_{xy}^2}$

$\sigma_1 := \sigma_m + \rho$

$\sigma_2 := \sigma_m - \rho$

$\theta_p := 0.5 \cdot \text{atan}\left(\frac{\tau_{xy}}{\sigma_d}\right)$

$\sigma_{check} := \sigma_m + \sigma_d \cdot \cos(2 \cdot \theta_p) + \tau_{xy} \cdot \sin(2 \cdot \theta_p)$

$\tau_{check} := -\sigma_d \cdot \sin(2 \cdot \theta_p) + \tau_{xy} \cdot \cos(2 \cdot \theta_p)$

$\sigma_1 = 5077.21$

$\sigma_2 = -324.21$

$\theta_p \cdot \frac{180}{\pi} = -14.18$

$\sigma_{check} = 5077.21$

$\tau_{check} = 0$

$\tau_1 := \rho$

$\tau_2 := -\rho$

$\theta_s := 0.5 \cdot \text{atan}\left(\frac{-\sigma_d}{\tau_{xy}}\right)$

$\tau_{check} := -\sigma_d \cdot \sin(2 \cdot \theta_s) + \tau_{xy} \cdot \cos(2 \cdot \theta_s)$

$\tau_1 = 2700.71$

$\tau_2 = -2700.71$

$\theta_s \cdot \frac{180}{\pi} = 30.82$

$\tau_{check} = -2700.71$

$\sigma_{normal} := \sigma_m$

$\sigma_{normal} = 2376.5$

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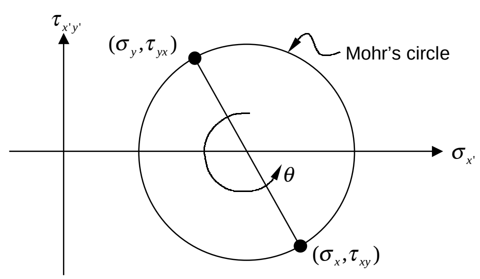
7.6 Mohr's Circle for Plane Stress

- Mohr's circle is a geometric interpretation of the plane stress transformation formulae:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

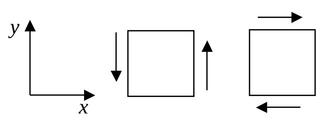
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

- We introduce a special sign convention for shear stresses -- this convention is used only for the construction of Mohr's circle:



Pairs of shear stresses:

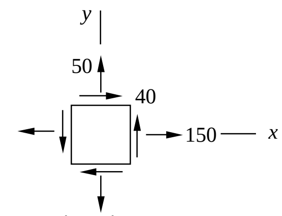
Clockwise is positive



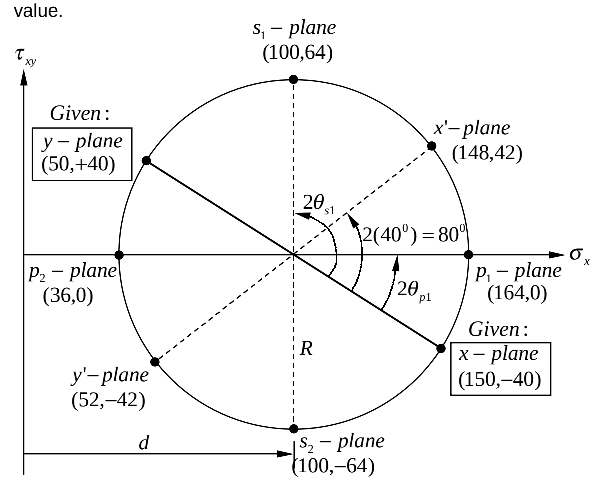
Stress tensor:	positive	positive
Mohr's circle:	negative	positive

Example 7.4

Given $\sigma_x = 150\,MPa,$
$\sigma_y = 50\,MPa,$
$\tau_{xy} = 40\,MPa$



Using Mohr's circle, determine:
(a) Determine the stresses on an element oriented at $\boldsymbol{\theta} = 40^0$
(b) The principal planes and principal stresses, and
(b) The planes of maximum shear stress and the maximum shear stress value.



$$d = \frac{\sigma_x + \sigma_x}{2} = 100 \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 64$$

- The coordinates of the required "planes" are as follows (computed from the circular geometry on the previous page):

$p_1 - plane: \quad \theta_{p1} = 19.3^0 \quad \sigma_1 = 164.0, \tau_{x'y'} = 0$

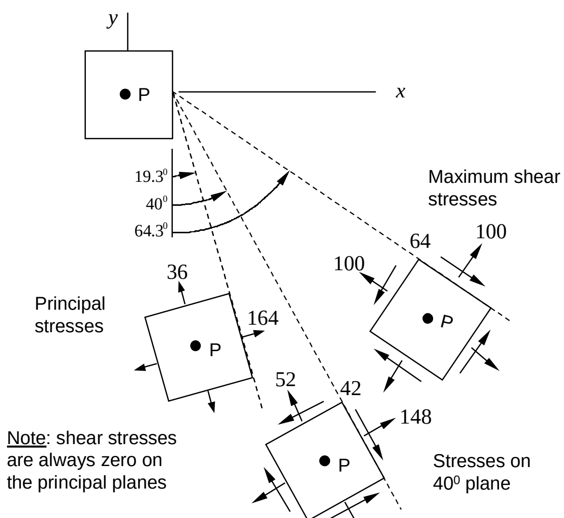
$p_2 - plane: \quad \theta_{p2} = 109.3^0 \quad \sigma_2 = 36.0, \tau_{x'y'} = 0$

$s_1 - plane: \quad \theta_{s1} = 64.3^0 \quad \sigma_{x'} = 100.0, \tau_1 = 64.0\,(CW)$

$s_2 - plane: \quad \theta_{s2} = 154.3^0 \quad \sigma_{x'} = 100.0, \tau_2 = -64.0\,(C-CW)$

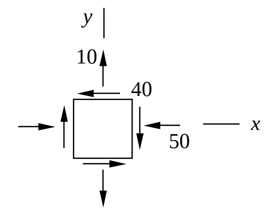
$40^0 - plane: \quad \theta = 40^0 \quad \sigma_{x'} = 148.0, \tau_{x'y'} = 42.0\,(CW)$

$130^0 - plane: \quad \theta = 130^0 \quad \sigma_{x'} = 52.0, \tau_{x'y'} = -42.0\,(C-CW)$

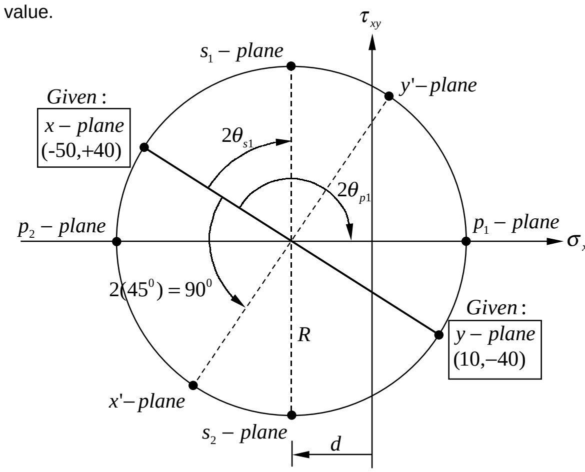


Example 7.5

Given $\sigma_x = -50\,MPa,$
$\sigma_y = 10\,MPa,$
$\tau_{xy} = -40\,MPa$



Using Mohr's circle, determine:
(a) Determine the stresses on an element oriented at $\theta = 45^0$
(b) The principal planes and principal stresses, and
(b) The planes of maximum shear stress and the maximum shear stress value.



$d = \frac{\sigma_x + \sigma_x}{2} = -20 \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 50$

- The coordinates of the required "planes" are as follows (computed from the circular geometry on the previous page):

$p_1 - plane: \quad \theta_{p1} = 63.4^0 \quad \sigma_1 = 30.0$

$p_2 - plane: \quad \theta_{p2} = 153.4^0 \quad \sigma_2 = -70.0$

$s_1 - plane: \quad \theta_{s1} = 18.4^0 \quad \sigma_{x'} = -20.0, \tau_1 = 50.0\,(CW)$

$s_2 - plane: \quad \theta_{s2} = 108.4^0 \quad \sigma_{x'} = -20.0, \tau_2 = -50.0\,(C-CW)$

$45^0 - plane: \quad \theta = 45^0 \quad \sigma_{x'} = -60.0, \tau_{x'y'} = -30.0\,(C-CW)$

$135^0 - plane: \quad \theta = 135^0 \quad \sigma_{x'} = 20.0, \tau_{x'y'} = 30.0\,(CW)$