Today's Topics

- Plane stress
- Stress transformation; Mohr's circle
- Principal stresses; maximum shear stress.

Reading Assignment

Problem Set #3  (Due October 18, 2000)

Juvinall
- 4.12
- 4.17
- 4.21
- 4.31
- 4.34

7.1 Stresses in Three Dimensions

- Stress at point is is expressed in terms of components with respect to a set of fixed coordinate axes.

- Consider the stress at a typical point $P$:

  - From equilibrium we know
    
    $\tau_{xy} = \tau_{yx}$,
    
    $\tau_{xz} = \tau_{zx}$,
    
    $\tau_{yz} = \tau_{zy}$

- There are 3 components of normal stress
  
  $\sigma_x, \sigma_y, \sigma_z$

- There are 3 components of shear stress
  
  $\tau_{xy}, \tau_{xz}, \tau_{yz}$
7.2 Stress in Two Dimensions (Plane Stress)

- Plane stress assumption:
  \( \sigma_{zz} = 0, \tau_{zx} = 0, \sigma_{yy} = 0 \)

- Positive stresses in plane stress

7.3 Transformation of Stress in Two Dimensions

- We need to understand how the components of the stress tensor change with rotation of coordinate axes.

**Given**

**Find**
By considering equilibrium, we can relate the components of the stress at P referred to both coordinate systems (see E14 notes and text):

\[
\sigma_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_y - \sigma_x}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

\[
\sigma_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_y - \sigma_x}{2} \cos 2\theta - \tau_{xy} \sin 2\theta
\]

\[
\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

Observe:

1. \(\sigma_y' = \sigma_x' (\text{at } \theta = \theta + 90^\circ)\)
   \[
   \sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_y - \sigma_x}{2} \cos 2\theta - \tau_{xy} \sin 2\theta
   \]

2. \(\sigma_x + \sigma_y = \sigma_x' + \sigma_y'\) -- called a stress invariant

Thought Experiment on Stress Transformation

(Why pure stretching also produces shearing)
Example 7.1

Given \( \sigma_x = 150 \text{ MPa}, \sigma_y = 50 \text{ MPa}, \tau_{xy} = 40 \text{ MPa} \)

Determine the stresses on an element oriented at \( \theta = 40^\circ \)

\[
\sigma_x = \frac{150 + 50}{2} + \frac{150 - 50}{2} \cos 80^\circ + 40 \sin 80^\circ = 148.1
\]

\[
\sigma_y = \frac{150 + 50}{2} - \frac{150 - 50}{2} \cos 80^\circ - 40 \sin 80^\circ = 51.9
\]

\[
\tau_{xy} = -\frac{150 - 50}{2} \sin 80^\circ + 40 \cos 80^\circ = -42.3
\]
7.4 Principal Stresses in 2-d

- There will exist values of $\theta$ for which the normal stress components are maximum (minimum):
  \[
  \frac{d\sigma}{d\theta} = 0 \Rightarrow \tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}
  \]
- Provides two angles $\theta_1$ and $\theta_2$ which differ by $90^\circ$ and define the principal planes.
- Using these angles in the expression for the transformed stress gives:
  \[
  \begin{align*}
  \theta_1 \rightarrow \sigma_x &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2} \\
  \theta_2 \rightarrow \sigma_y &= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}
  \end{align*}
  \]
- Note that:
  1. Shear stress on the principal planes is zero.
  2. $\sigma_1 > \sigma_2$.
- **Question** -- Given the two roots of $\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$, how will you decide which angle defines which principal plane?
- **Answer** -- take either root and substitute it into:
  \[
  \sigma_x = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_2 - \sigma_1}{2} \cos 2\theta + \tau \sin 2\theta
  \]
  and compare $\sigma_x$ with the known values of $\sigma_1$ and $\sigma_2$.

7.5 Maximum Shear Stresses

- There will exist values of $\theta$ for which the shear stress components are maximum (minimum):
  \[
  \frac{d\tau}{d\theta} = 0 \Rightarrow \tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau}
  \]
- Provides two angles $\theta_1$ and $\theta_2$ which differ by $90^\circ$ and define the principal planes.
- Using these angles in the expression for the transformed stress gives:
  \[
  \begin{align*}
  \theta_1 \rightarrow \tau_{x'y'} &= +\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2} \\
  \theta_2 \rightarrow \tau_{x'y'} &= -\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}
  \end{align*}
  \]
- Note that:
  1. $\tau_{\text{max}} = \frac{\sigma_x - \sigma_y}{2}$ (useful result!)
  2. Mean stress on both planes of max/min shear stress is:
  \[
  \sigma_{x'y'} = \frac{\sigma_1 + \sigma_2}{2}
  \]
  3. Observe $\sigma_z = \sigma_1 + \sigma_2$. 

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Example 7.2

Given \( \sigma_y = 150 \text{ MPa}, \ \sigma_x = 50 \text{ MPa}, \ \tau_{xy} = 40 \text{ MPa} \)

Determine:
(a) the principal planes and principal stresses, and
(b) the planes of maximum shear stress and the maximum shear stress value.

(a) Principal planes and stresses:

\[
\sigma_{1,2} = \frac{\sigma_y + \sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}
\]

\[
= \frac{150 + 50}{2} \pm \sqrt{\left(\frac{150 - 50}{2}\right)^2 + 40^2}
\]

= 164.0, 36.0

Now let's find the principal planes:

\[
tan 2\theta = \frac{2\tau_{xy}}{\sigma_y - \sigma_x} \quad \Rightarrow \quad 2\theta = 38.66^\circ, 38.66^\circ \pm 180^\circ
\]

\[
\Rightarrow \quad \theta_1 = 19.33^\circ, 19.33^\circ \pm 90^\circ
\]

Consider \( \theta_1 = 19.33^\circ \)

\[
\sigma_y = \frac{\sigma_y + \sigma_x}{2} + \frac{\sigma_y - \sigma_x}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

\[
= \frac{150 + 50}{2} + \frac{150 - 50}{2} \cos 2(19.33^\circ) + 40 \sin 2(19.33^\circ)
\]

\[
= 164.0
\]

\[
\Rightarrow \quad \theta_{\sigma_1} = 19.33^\circ, \quad \theta_{\sigma_2} = 109.33^\circ
\]

(b) Maximum shear stresses:

\[
\tau_{\text{max}} = \pm \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2} \pm \sqrt{\left(\frac{150 - 50}{2}\right)^2 + 40^2}
\]

\[
= \pm 64.0
\]

Now let's find the maximum shear planes:

\[
tan 2\theta = \frac{\sigma_y - \sigma_x}{2\tau_{xy}} \quad \Rightarrow \quad 2\theta = -51.34^\circ, -51.34^\circ \pm 180^\circ
\]

\[
\Rightarrow \quad \theta_1 = -25.67^\circ, -25.67^\circ \pm 90^\circ
\]

Consider \( \theta_1 = -25.67^\circ \)

\[
\tau_{\text{max}} = -\frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

\[
= -\frac{150 - 50}{2} \sin 2(-25.67^\circ) + 40 \cos 2(-25.67^\circ)
\]

\[
= 64.0
\]

\[
\Rightarrow \quad \theta_{\tau_{\text{max}}} = -25.67^\circ, \quad \theta_{\tau_{\text{max}}} = 64.33^\circ
\]

Normal stress = \( \frac{\sigma_y + \sigma_x}{2} = \frac{150 + 50}{2} = 100 \)

Finally, let's check that:

\[
\tau_{\text{max}} = \frac{\sigma_y - \sigma_x}{2} \quad \Rightarrow \quad + 64 = \frac{164 - 36}{2} = \text{true}
\]
Example 7.3

Returning to Example X.X, determine the principal stresses and maximum shear stresses at points (a) and (b).

From example X.X, we have:

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_x$</td>
<td>$-7.319$</td>
<td>$4.753$</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\tau_{xy}$</td>
<td>$-1.434$</td>
<td>$-1.283$</td>
</tr>
</tbody>
</table>
Point (a)

\[ \sigma_x = -7319 \]
\[ \sigma_y = 0 \]
\[ \tau_{xy} = -1434 \]

\[ \sigma_m = \frac{(\sigma_x + \sigma_y)}{2} \]
\[ \sigma_d = \frac{(\sigma_x - \sigma_y)}{2} \]
\[ \rho = \sqrt{\sigma_d^2 + \tau_{xy}^2} \]
\[ \sigma_1 = \sigma_m + \rho \]
\[ \sigma_2 = \sigma_m - \rho \]
\[ \theta_p = 0.5 \text{ atan} \left( \frac{\tau_{xy}}{\sigma_d} \right) \]
\[ \sigma_{\text{check}} = \sigma_m + \sigma_d \cos \left( 2 \theta_p \right) + \tau_{xy} \sin \left( 2 \theta_p \right) \]
\[ \tau_{\text{check}} = -\sigma_d \sin \left( 2 \theta_p \right) + \tau_{xy} \cos \left( 2 \theta_p \right) \]
\[ \sigma_1 = 270.93 \]
\[ \sigma_2 = 7589.93 \]
\[ \theta_p \left( \frac{180}{\pi} \right) = 10.7 \]
\[ \sigma_{\text{check}} = -7589.93 \]
\[ \tau_{\text{check}} = 0 \]

\[ \sigma_{\text{normal}} = \sigma_m \]
\[ \sigma_{\text{check}} = 7589.93 \]
\[ \tau_{\text{check}} = 0 \]

Point (b)

\[ \sigma_x = 4753 \]
\[ \sigma_y = 0 \]
\[ \tau_{xy} = -1283 \]

\[ \sigma_m = \frac{(\sigma_x + \sigma_y)}{2} \]
\[ \sigma_d = \frac{(\sigma_x - \sigma_y)}{2} \]
\[ \rho = \sqrt{\sigma_d^2 + \tau_{xy}^2} \]
\[ \sigma_1 = \sigma_m + \rho \]
\[ \sigma_2 = \sigma_m - \rho \]
\[ \theta_p = 0.5 \text{ atan} \left( \frac{\tau_{xy}}{\sigma_d} \right) \]
\[ \sigma_{\text{check}} = \sigma_m + \sigma_d \cos \left( 2 \theta_p \right) + \tau_{xy} \sin \left( 2 \theta_p \right) \]
\[ \tau_{\text{check}} = -\sigma_d \sin \left( 2 \theta_p \right) + \tau_{xy} \cos \left( 2 \theta_p \right) \]
\[ \sigma_1 = 5077.21 \]
\[ \sigma_2 = 324.21 \]
\[ \theta_p \left( \frac{180}{\pi} \right) = 30.82 \]
\[ \sigma_{\text{check}} = 5077.21 \]
\[ \tau_{\text{check}} = 0 \]

\[ \sigma_{\text{normal}} = \sigma_m \]
\[ \sigma_{\text{normal}} = 2376.5 \]
7.6 Mohr’s Circle for Plane Stress

• Mohr’s circle is a geometric interpretation of the plane stress transformation formulae:

\[
\sigma_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

\[
\tau_{xy} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

• We introduce a special sign convention for shear stresses -- this convention is used only for the construction of Mohr’s circle:

\[
\tau_{xy} = \begin{cases} 
+ \tau_{xy} & \text{clockwise} \\
- \tau_{xy} & \text{counterclockwise}
\end{cases}
\]

Example 7.4

Given \( \sigma_x = 150 \text{ MPa}, \)  
\( \sigma_y = 50 \text{ MPa}, \)
\( \tau_{xy} = 40 \text{ MPa} \)

Using Mohr’s circle, determine:
(a) Determine the stresses on an element oriented at \( \theta = 40^\circ \)
(b) The principal planes and principal stresses, and
(b) The planes of maximum shear stress and the maximum shear stress value.

Stress tensor: positive positive  
Mohr’s circle: positive negative positive

Pairs of shear stresses:
Clockwise is positive
The coordinates of the required “planes” are as follows (computed from the circular geometry on the previous page):

- Plane 1: \( p_1 \): \( \theta_{p_1} = 19.3^\circ \), \( \sigma_x = 164.0 \), \( \tau_{x'y'} = 0 \)
- Plane 2: \( p_2 \): \( \theta_{p_2} = 109.3^\circ \), \( \sigma_x = 36.0 \), \( \tau_{x'y'} = 0 \)
- Plane 3: \( s_1 \): \( \theta_{s_1} = 64.3^\circ \), \( \sigma_x = 100.0 \), \( \tau_x = 64.0 \) (CW)
- Plane 4: \( s_2 \): \( \theta_{s_2} = 154.3^\circ \), \( \sigma_x = 100.0 \), \( \tau_x = -64.0 \) (C - CW)
- Plane 5: \( 40^\circ \): \( \theta = 40^\circ \), \( \sigma_x = 148.0 \), \( \tau_{x'y'} = 42.0 \) (CW)
- Plane 6: \( 130^\circ \): \( \theta = 130^\circ \), \( \sigma_x = 52.0 \), \( \tau_{x'y'} = -42.0 \) (C - CW)

Example 7.5

Given: \( \sigma_x = -50 \text{ MPa} \),
\( \sigma_y = 10 \text{ MPa} \),
\( \tau_{xy} = -40 \text{ MPa} \)

Using Mohr’s circle, determine:
(a) Determine the stresses on an element oriented at \( \theta = 45^\circ \)
(b) The principal planes and principal stresses, and
(b) The planes of maximum shear stress and the maximum shear stress value.

Note: shear stresses are always zero on the principal planes.
• The coordinates of the required “planes” are as follows (computed from the circular geometry on the previous page):
  
  \( p_1 \text{ plane}: \quad \theta_{p1} = 63.4^0, \quad \sigma_i = 30.0 \)
  
  \( p_2 \text{ plane}: \quad \theta_{p2} = 153.4^0, \quad \sigma_i = -70.0 \)
  
  \( s_1 \text{ plane}: \quad \theta_{s1} = 18.4^0, \quad \sigma_s = -20.0, \tau_s = 50.0 \text{ (CW)} \)
  
  \( s_2 \text{ plane}: \quad \theta_{s2} = 108.4^0, \quad \sigma_s = -20.0, \tau_s = -50.0 \text{ (C-CW)} \)
  
  \( 45^0 \text{ plane}: \quad \theta = 45^0, \quad \sigma_x = -60.0, \tau_{xy} = -30.0 \text{ (C-CW)} \)
  
  \( 135^0 \text{ plane}: \quad \theta = 135^0, \quad \sigma_x = 20.0, \tau_{xy} = 30.0 \text{ (CW)} \)