I certify that I upheld the Stanford Honor code during this exam

- Print your name and sign the honor code statement
- You may use your course notes, homework, books, etc.
- Write your answers on this handout
- Where space is provided, show your work to get credit
- If necessary, attach extra pages for scratch work
- Best wishes for a fun vacation. Merry Christmas and Happy New Year!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Value</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td></td>
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<tr>
<td>3</td>
<td>31</td>
<td></td>
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<tr>
<td>4</td>
<td>20</td>
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<td>5</td>
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<td>6</td>
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<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Final 1 (6 pts.) Miscellaneous

(a) (1 pt.) Describe one or two things you enjoyed learning in ME161.
   Result:
(b) (1 pt.) Name the engineer, scientist, or mathematician who first did the following:

- Considered sine and cosine as functions (not just the ratio of two sides of a triangle)
- Invented the symbols $\pi = 3.1415..., \ e = 2.71..., \ i = \sqrt{-1}, \ \sum$ for summations, $\Delta x$ for finite differences, and $f(t)$ for functions
- Discovered a general equation for rotational motions of a rigid body
- Created the foundations of the theory of differential equations
- Discovered a simple technique for numerically integrating differential equations
- Investigated analytical functions of a complex variable
- Invented the Taylor series expansion of a function (concurrent with MacLaurin)
- Studied mathematics under Johann Bernoulli at 14 years old
- Extended Bernoulli's equation for incompressible fluid flow
- Optimized the arrangement of masts on a ship
- "Wrote the book" on the mathematical theory of music
- Worked on cartography, magnetism, fire engines, ship building, and insurance
- Lost 8 of his 13 children in infancy, went half blind at 28 and mostly blind at 59, lost his home to a fire, and still had a positive attitude

(c) Any real, imaginary, or complex number can be expressed in magnitude-phase form.

- (1 pt.) Express $-2$ in magnitude-phase form.
  $$-2 = e^{(-2n\pi)i} \quad n=0,1,2,...$$

- (1 pt.) Why does multiplying two negative numbers produce a positive number?
  Using magnitude-phase form, show $-2 \times -2 = +4$.

- (2 pts.) Complex numbers and exponentiation
  Find all complex numbers (in Cartesian form) equal to the following.
  $$\sqrt{i} = _____ + _____ i \quad \text{or} \quad _____ + _____ i$$

  $$1^{\frac{1}{2\pi}} = 1,$$
Final 2 (13 pts.) **Sinusoidal transfer function and underdamped vibrations**

A dynamic system’s response is governed by $\ddot{y} + \dot{y} + y = f(t)$ where $f(t) = 10 \sin(2t)$.

(a) **(2 pts.)** Find the system’s transfer function

\[ G(s) = \frac{Y(s)}{F(s)} = \]

(b) **(3 pts.)** Is the transfer function stable? **Yes/No**.

Explain:

(c) **(5 pts.)** Find numerical values for the magnitude and phase of the sinusoidal transfer function.

\[ \text{Magnitude} = \quad \text{Phase} = \quad \text{rad} \]

(d) **(3 pts.)** Find the steady-state response $y_{ss}(t)$.

\[ y_{ss}(t) = \]
Final.3 (31 pts.) Solution and stability of a tight-rope walker
A tightrope walker \( A \) uses a rigid pole \( B \) to balance on a wire at a point \( N_0 \) that is fixed in a Newtonian reference frame \( N \). Right-handed sets of orthogonal unit vectors are fixed in \( N, A, \) and \( B \), with:

- \( \mathbf{n}_x \) horizontal and to the right
- \( \mathbf{n}_y \) vertically upward
- \( \mathbf{a}_y \) directed from \( N_0 \) to \( S_c \) (the mass center of \( A \) and \( B \))
- \( \mathbf{b}_x \) directed along the balancing pole
- \( \mathbf{n}_z = \mathbf{a}_z = \mathbf{b}_z \) perpendicular to the plane in which \( A \) and \( B \) move

The following identifiers are useful in this analysis.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of system formed by ( A ) and ( B )</td>
<td>( m )</td>
<td>constant</td>
</tr>
<tr>
<td>Distance from ( N_0 ) to ( S_c )</td>
<td>( d )</td>
<td>constant</td>
</tr>
<tr>
<td>Related to mass distribution of system</td>
<td>( I )</td>
<td>constant</td>
</tr>
<tr>
<td>Central moment of inertia of ( B ) for ( \mathbf{b}_x )</td>
<td>( I^B )</td>
<td>constant</td>
</tr>
<tr>
<td>Earth’s sea-level gravitational constant</td>
<td>( g )</td>
<td>constant</td>
</tr>
<tr>
<td>Feedback-control torque on ( B ) from ( A )</td>
<td>( T_z )</td>
<td>specified variable</td>
</tr>
<tr>
<td>Angle between ( \mathbf{n}_y ) and ( \mathbf{a}_y )</td>
<td>( \theta_A )</td>
<td>dependent variable</td>
</tr>
<tr>
<td>Angle between ( \mathbf{n}_x ) and ( \mathbf{b}_x )</td>
<td>( \theta_B )</td>
<td>dependent variable</td>
</tr>
<tr>
<td>Time</td>
<td>( t )</td>
<td>independent variable</td>
</tr>
</tbody>
</table>

(a) (1 pt.) The nonlinear equations of motion for the tightrope walker are

\[
I \ddot{\theta}_A + mgd \sin(\theta_A) = -T_z \\
I^B \ddot{\theta}_B = T_z
\]

Determine \( M, K, \) and \( G \) so that the linearized ODEs can be written in the matrix form

\[
M \ddot{X} + KX = GT_z \text{ where } X = \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}.
\]

Result:

\[
\begin{bmatrix}
\ddot{\theta}_A \\
\ddot{\theta}_B
\end{bmatrix} + \begin{bmatrix}
\quad \\
\quad
\end{bmatrix}
= \begin{bmatrix}
\theta_A \\
\theta_B
\end{bmatrix} = \begin{bmatrix}
\quad \\
\quad
\end{bmatrix} T_z
\]

(b) (3 pts.) Consider the homogeneous problem \( T_z = 0 \). Assume a solution \( X(t) = U e^{pt} \) where \( p \) is a constant (to-be-determined) and \( U \) is a non-zero \( 2 \times 1 \) matrix of constants (to-be-determined). Substitute this assumed solution into the matrix equation and find the equation that governs \( p \) and \( U \). Express your results in terms of \( \lambda \triangleq -p^2, A \triangleq M^{-1} K, \) and the \( 2 \times 2 \) identity matrix \( I \).

Result:

\[
= 0
\]

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(c) (4 pts.) Calculate the $2 \times 2$ matrix $A = M^{-1} K$ in terms of $m, g, d, I^B$ and $I$.

Result:

$$A = \begin{bmatrix} \end{bmatrix}$$

(d) (4 pts.) Calculate $\lambda_i (i = 1, 2)$.

Result:

$$\lambda_1 = \quad \lambda_2 =$$

(e) (2 pts.) Find $p_1, p_2, p_3$ and $p_4$.

Result:

$$p_1 = \quad p_3 =$$

$$p_2 = \quad p_4 =$$

(f) (2 pts.) The solution for $X(t) = \begin{bmatrix} \theta_A(t) \\ \theta_B(t) \end{bmatrix}$ is stable/neutrally stable/unstable.

(g) (2 pts.) A larger value of $m$ corresponds to a less/more stable solution.

A larger value of $I$ corresponds to a less/more stable solution.
(h) **(6 pts.)** Find $U_1$ and $U_2$, the eigenvectors that correspond to $\lambda_1$ and $\lambda_2$.

Result:

\[
U_1 = \begin{bmatrix} \phantom{c} \\ \phantom{c} \\
\end{bmatrix} \quad U_2 = \begin{bmatrix} \phantom{c} \\ \phantom{c} \\
\end{bmatrix}
\]

(i) **(2 pts.)** Assemble the solution for $X(t)$

Result:

\[
\begin{bmatrix}
\theta_A(t) \\
\theta_B(t)
\end{bmatrix} = c_1 \begin{bmatrix} \phantom{c} \\
\phantom{c} \\
\end{bmatrix} + c_2 \begin{bmatrix} \phantom{c} \\
\phantom{c} \\
\end{bmatrix} + c_3 \begin{bmatrix} \phantom{c} \\
\phantom{c} \\
\end{bmatrix} + c_4 \begin{bmatrix} \phantom{c} \\
\phantom{c} \\
\end{bmatrix}
\]

(j) **(1 pt.)** How are the constants $c_1, c_2, c_3, c_4$ usually determined?

(k) **(4 pts.)** Sketch the system moving in its two modes. Clearly show how $\theta_A$ and $\theta_B$ are changing.

---

Moving in Mode 1

Moving in Mode 2
Final 4 (20 pts.) State-space feedback control of a tight-rope walker

One way to design an automatic control system for the tight-rope walker is to use the state-space method. The state space-method begins by defining the state matrix \( Y \) as

\[
Y \triangleq \begin{bmatrix}
\theta_A \\
\theta_B \\
\dot{\theta}_A \\
\dot{\theta}_B 
\end{bmatrix}
\]

(a) (1 pt.) Suppose the linearized ODEs for the tight-rope walker are

\[100 \ddot{\theta}_A + 500 \theta_A = -T_z \]
\[25 \ddot{\theta}_B = T_z \]

Solve for \( \ddot{\theta}_A \) and \( \ddot{\theta}_B \) in terms of \( \theta_A, \theta_B, T_z \), etc.,

Result:

\[
\ddot{\theta}_A =
\]
\[
\ddot{\theta}_B =
\]

(b) (4 pts.) Cast these ODEs into the state-space form \( \dot{Y} = AY + B_cT_z \) by completing the following matrices.

Result:

\[
\begin{bmatrix}
\dot{\theta}_A \\
\dot{\theta}_B \\
\dot{\theta}_A \\
\dot{\theta}_B 
\end{bmatrix} =
\begin{bmatrix}
\theta_A \\
\theta_B \\
\dot{\theta}_A \\
\dot{\theta}_B 
\end{bmatrix} +
\begin{bmatrix}
A \\
B \\
C \\
D 
\end{bmatrix} T_z
\]

(c) (4 pts.) An engineer decides to balance the tight-rope walker using a feedback control law for \( T_z \) that is written in terms of the “feedback control constants” \( k_1, k_2, k_3, k_4 \), as

\[T_z = k_1 \theta_A + k_2 \theta_B + k_3 \dot{\theta}_A + k_4 \dot{\theta}_B = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} \theta_A \\
\theta_B \\
\dot{\theta}_A \\
\dot{\theta}_B 
\end{bmatrix} = +K_cY \]

Rewrite the ODEs in the state-space form \( \dot{Y} = AY \) in terms of \( k_1, k_2, k_3, k_4 \), and numbers.

Result:

\[
\dot{Y} = AY
\]

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(d) (1 pt.) Using physical intuition, guess at the signs of \(k_1, k_2, k_3, \) and \(k_4\) so that if you started with \(\theta_A=10^5\) and \(\theta_B=0^\circ\), you could bring this system to rest with \(\theta_A=0^\circ\) and \(\theta_B=0^\circ\). Circle — if you think the number is negative, 0 if you think the number is zero, or + if you think the number is positive. In other words, should the coefficients of \(\theta_A, \theta_B, \dot{\theta}_A, \) and \(\dot{\theta}_B\) be negative, zero, or positive in order to bring this system to rest at \(\theta_A = \theta_B = 0^\circ\).

Result:
\[
T_z = \left( + 0 - \right) \theta_A + \left( + 0 - \right) \theta_B + \left( + 0 - \right) \dot{\theta}_A + \left( + 0 - \right) \dot{\theta}_B
\]

(e) (2 pts.) The process of solving for \(Y(t)\) begins by assuming a solution of the form \(Y(t) = U e^{\lambda t}\) where \(\lambda\) is a constant (to be determined), and \(U\) is a non-zero \(4 \times 1\) matrix of constants (to be determined). After substituting this assumed solution into the governing ODE, the matrix equation that governs \(\lambda\) and \(U\) is
\[
\left(-\lambda I + A\right) U = 0
\]

The unknowns in the previous equation are \(\lambda\) and \(U\). Classify the previous matrix equation by picking the relevant qualifiers from the following list.

<table>
<thead>
<tr>
<th>Uncoupled</th>
<th>Linear</th>
<th>Homogeneous</th>
<th>Algebraic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupled</td>
<td>Nonlinear</td>
<td>Inhomogeneous</td>
<td>Differential</td>
</tr>
</tbody>
</table>

(f) (2 pts.) How does one solve for \(\lambda\)?

(g) (3 pts.) The polynomial equation that relates \(\lambda\) to \(k_1, k_2, k_3, \) and \(k_4\) is (you do not need to show this)
\[
\lambda^4 + (0.01 k_3 - 0.04 k_4) \lambda^3 + (0.01 k_1 - 0.04 k_2 - 5) \lambda^2 + (0.4 k_4) \lambda + (0.4 k_2) = 0
\]

Determine whether \(k_i\) (i=1,2,3,4) must be negative, zero, positive, or undetermined for the roots of \(\lambda\) to have negative real parts so that \(Y = U e^{\lambda t}\) is stable.

Result:

<table>
<thead>
<tr>
<th>Feedback Control Constant</th>
<th>Negative, zero, positive, or undetermined</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_1)</td>
<td>negative/zero/positive/undetermined</td>
</tr>
<tr>
<td>(k_2)</td>
<td>negative/zero/positive/undetermined</td>
</tr>
<tr>
<td>(k_3)</td>
<td>negative/zero/positive/undetermined</td>
</tr>
<tr>
<td>(k_4)</td>
<td>negative/zero/positive/undetermined</td>
</tr>
</tbody>
</table>

(h) (1 pt.) Since the feedback control law is \(T_z = k_1 \theta_A + k_2 \theta_B + k_3 \dot{\theta}_A + k_4 \dot{\theta}_B\) and you know some (or all) of the signs of \(k_1, k_2, k_3, \) and \(k_4\), circle the correct signs in the following equation.

Result:
\[
T_z = \left( + 0 - \right) \theta_A + \left( + 0 - \right) \theta_B + \left( + 0 - \right) \dot{\theta}_A + \left( + 0 - \right) \dot{\theta}_B
\]

(i) (2 pts.) Determine the values of \(\lambda\) that satisfy its polynomial equation when the system is uncontrolled, i.e., \(k_1 = k_2 = k_3 = k_4 = 0\).

Result:
\[
\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 =
\]

(j) (1 pts.) The uncontrolled system is stable/neutrally stable/unstable.
Final 5 (6 pts.) Cruise control for a car
A model for the speed \( v \) of a car of mass \( m \) includes unmodeled forces \( F_{\text{disturbance}} \), an air-resistance drag force \( F_{\text{Drag}} \), and the control force \( F_c \) (exerted by the engine and wheels) that tries to move the car at a desired (nominal) speed \( v_{\text{nom}} \).

(a) (2 pts.) Determine \( F_{\text{nom}} \), the value of \( F_c \) required so \( v = v_{\text{nom}}(t) \) when \( F_{\text{disturbance}} = 0 \).

\[
F_{\text{nom}} =
\]

(b) (2 pts.)
After separating \( F_c \) into two terms as shown to the right, rewrite the ODE in terms of \( \bar{v} \) [the error between the actual value of \( v \) and the desired (nominal) value of \( v \)].

\[
F_c = F_{\text{nom}} + \bar{F}_c
\]

\[
\bar{v} = v - v_{\text{nom}}
\]

(c) (2 pts.) What advantage does this model that includes air-resistance (drag) have over a model that does not include air-resistance? Explain.

Final 6 (15 pts.) Proportional (P) feedback control for a simple car model
One choice for \( \bar{F}_c \) is a proportional control law of the form

\[
\bar{F}_c = -k_p \cdot \bar{v}
\]

where \( k_p \) is a constant.

(a) (7 pts.) Assuming \( F_{\text{disturbance}} \) is constant, solve for \( \bar{v}(t) \) in terms of \( F_{\text{disturbance}}, m, k_p, t, \) and the initial error \( \bar{v}(0) \).

\[
\bar{v}(t) =
\]
(b) (1 pt.) When \( F_{\text{disturbance}} \) is constant, the steady-state error is \( \bar{v}_{ss} = \)

(c) (1 pt.) Making \( k_p \) small/large & negative/positive gives the most stable solution for \( \bar{v}(t) \)

(d) (6 pts.) Find the values of \( k_p \) that satisfy the following specifications for a 1000 kg car.

<table>
<thead>
<tr>
<th>Design specification</th>
<th>( k_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>The steady-state error in response to ( F_{\text{disturbance}} = 800 \text{ n} ) is ( -2 \frac{\text{m}}{\text{sec}} )</td>
<td></td>
</tr>
<tr>
<td>With ( F_{\text{disturbance}} = 0 ), the car goes from rest to 90% of its desired speed within 5 sec</td>
<td></td>
</tr>
</tbody>
</table>

Final.7 (9 pts.) Proportional-Integral (PI) cruise control for a simple car model

One choice for \( \bar{F}_c \) is a **proportional-integral control law** \( \bar{F}_c = -k_p * \bar{v} + -k_i * \int_{t=0}^{t} \bar{v} d\bar{F} \) where \( k_p \) and \( k_i \) are constants.

(a) (3 pts.) Write a 2\textsuperscript{nd}-order, inhomogeneous, ODE in standard form for \( \bar{v}(t) \).

(\( F_{\text{disturbance}} \) is constant).

(b) (1 pt.) When \( F_{\text{disturbance}} \) is constant, the steady-state error is \( \bar{v}_{ss} = \)

(c) (5 pts.) Find values of \( k_p \) and \( k_i \) that satisfy the following specifications for a 1000 kg car.

<table>
<thead>
<tr>
<th>Design specification</th>
<th>( k_p )</th>
<th>( k_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>With ( F_{\text{disturbance}} = 0 ), the car goes from rest and settles to within 1% of its desired speed within 5 sec with a maximum overshoot of 2%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>