4 Lab: Forcing functions for a 2\textsuperscript{nd}-order, linear, ODE

Some \textit{inputs} to dynamic systems result in interesting dynamic \textit{response}. Inputs that excite dynamic systems include periodic inputs (such as a vibrating motor) or one-time disturbances (such as the step displacement or impulse used in previous labs to determine system parameters). This lab investigates \textit{harmonic} (periodic) sinusoidal inputs because

- A rotary motor with an offset mass creates a sinusoidal forcing function
- The 1\textsuperscript{st}-term of the Fourier approximation dominates many real periodic forcing functions

The input forcing function is created by a motor with an offset mass. The motor’s base is rigidly attached to a cart that moves along a flat horizontal track. The ODE that describes the horizontal displacement $x$ of the cart is similar to the ODEs associated with the Scotch-yoke and air-conditioner homework problems. The input motor’s angular speed $\Omega$ will be measured by an encoder. The cart’s horizontal acceleration $\ddot{x}$ will be measured using an accelerometer. You will see the affect of $\Omega$ on $x$ and $\ddot{x}$.

4.1 PreLab: Working Model and brainstorming

1. Download the following Working Model simulations from the class website:
   HarmonicForcingPogoStick.wm2d
2. Run the Working Model simulations.
   Record results on the Working Model PreLab (see back of the book).
3. Complete the in-class Working Model PreLab problem on Harmonic forcing of a mass-spring-damper system
4. Complete homework problem \textit{Harmonic Forcing of a mass-spring-damper system}, parts a, b, c
5. The most difficult parameter to physically measure is $\frac{m}{b/k}$ (circle one).
   Explain: 
6. Loosely speaking, natural frequency $\omega_n$ is __________.
7. The magnitude of the output response is relatively \textit{small/large} when $\Omega \ll \omega_n$
8. The magnitude of the output response is relatively \textit{small/large} when $\Omega \approx \omega_n$
9. The magnitude of the output response is relatively \textit{small/large} when $\Omega \gg \omega_n$
4.2 PreLab: Pulse Width Modulation (PWM)

To control a motor, it is often desirable to power the motor using a wide range of voltage values – even though many power supplies have one constant voltage. One way to overcome this obstacle is with Pulse Width Modulation (PWM). In PWM, the micro-controller turns the motor on and off at a specified frequency, varying the ratio between the On and Off periods to control the average voltage provided to the motor. For example, the micro-controller can quickly vary the voltage to the motor from 0 volts (Off) to 12 volts (On) for equal amounts of time so it seems the motor receives 6 Volts.

PWM is controlled by a **duty cycle**, defined as the ratio of time the signal is On to the total period (total time the voltage is On or Off) – hence it has values between 0 and 1. The average voltage to the motor is calculated as

\[ V_{\text{out}} = \text{DutyCycle} \times V_{\text{supply}} \]

For the Arduino micro-controller, duty cycle is selected as an integer between 0 and 255, with 0 corresponding to always Off (0 Volts) and 255 corresponding to always On (12 Volts), hence

\[ \text{DutyCycle} = \frac{\text{ArduinoIntegerValue}}{255} \]

Knowing the power supply provides \( V_{\text{supply}} \approx 12 \text{ Volts} \), calculate the corresponding Arduino integer value to generate the following desired output voltages.

**Result:**

\[ V_{\text{out}} = 2.5 \text{ Volts} \quad \text{ArduinoIntegerValue} = \text{___} \]

\[ V_{\text{out}} = 7 \text{ Volts} \quad \text{ArduinoIntegerValue} = \text{___} \]

Knowing \( V_{\text{out}} \) has 255 discrete values, calculate the smallest voltage increment \( \Delta V_{\text{out}} \).

**Result:**

\[ \Delta V_{\text{out}} = \text{___} \text{ Volts} \]

4.3 Experimental

The first step in solving an **inhomogeneous** ODE, is to solve the **homogeneous** part of the ODE. Similarly, the first step in determining a system’s **forced response** is to determine the system’s **unforced** (natural) response. This lab uses hardware similar to Lab 2 (mass/spring cart system) but employs an offset-mass motor that is rigidly attached to the cart.

4.3.1 Determination of physical and electrical parameters

Determine the natural frequency \( \omega_n \) and damping ratio \( \zeta \) for horizontal motions \( x \) of the cart **with** the motor attached to it. Use Part 1 of the Arduino Lab4.ino program to capture necessary data (described in Section 4.3.5, steps 2-8). Hint: Remember Lab 2.

\[ \omega_n = \text{___} \frac{\text{rad}}{\text{sec}} \quad \zeta = \text{___} \text{ NoUnits} \]
Record the A/D steady-state value of the accelerometer and accompanying voltage offset $V_{0g}$ when the cart is not moving. These values will be used later when calculating the magnitude of the accelerometer’s signal. Hint: 5 Volts = 1024 AD Ticks

Result:

$$A/D_{\text{SteadyState}} = \boxed{\text{Volts}} \quad V_{0g} = \boxed{\text{Volts}}$$

4.3.2 Analytical prediction

The steady-state solution (i.e., the long-term behavior) of the stable, constant-coefficient, linear, 2nd-order ODE that governs this sinusoidally-forced laboratory system is

$$x_{ss}(t) = B \sin(\Omega t + \phi) \quad \text{where} \quad B = \frac{m_{\text{offset}}r}{(m_{\text{offset}} + m_{\text{cart}}) \sqrt{\left[\left(\frac{\omega_n}{\Omega}\right)^2 - 1\right]^2 + [2 \zeta \left(\frac{\omega_n}{\Omega}\right)]^2}}$$

Determine $\ddot{x}_{ss}$ and its magnitude $|\ddot{x}_{ss}|$ in terms of $B$, $\Omega$, $t$, and/or $\phi$.

On the following graph,\(^8\) plot $|x_{ss}(t)|$ for $0 \leq \Omega \leq 2\omega_n$ (use your experimentally-determined values of $\omega_n$ and $\zeta$). Similarly, plot $|\ddot{x}_{ss}(t)|$ versus $\Omega$. Label the scale on the left-axis appropriately for the range of values of $|x_{ss}(t)|$ and label the scale of the right-axis for the range of values of $|\ddot{x}_{ss}(t)|$. (Note: The left-scale and right-scale will be different). Please show your plots to a TA before moving on.

\(^8\)You may use Excel, MATLAB\textsuperscript{®}, or PlotGenesis to plot the graphs. Ensure the scales on both the left-axis and right-axis are both labeled.
4.3.3 Experimental estimation with an accelerometer

We use several pieces of equipment to measure and record the cart’s horizontal acceleration, namely, we use an accelerometer, a microprocessor, a transceiver, and a computer.  

- **Accelerometer: ADXL 311 from Analog Devices**
  The accelerometer is mounted on a cart and measures acceleration in up to three directions (we use data from only one direction). The accelerometer is relatively small and lightweight as compared to the cart - so its affect on the acceleration of the cart is negligible. The accelerometer’s output signal is a linear $0.3 \, \frac{\text{volts}}{g}$ signal over a range of $\pm \, 2 \, g$. The accelerometer is designed to output 2.5 volts when there is no acceleration, but there is some variation from one accelerometer to the next. This is why we measured the output of the accelerometer in Section 4.3.1.

\[
v_{\text{accelerometer}} \approx v_{bg} + \left( 0.3 \, \frac{\text{volts}}{1 \, g} \right) \ddot{x}
\]

- **Breadboard signal processing**
  The accelerometer’s signal is filtered by a low-pass filter to remove high-frequency noise in the signal. Op-amps are used as a buffer to supply additional current and avoid unwanted voltage drops.

- **Arduino UNO microprocessor:**
  The microprocessor’s A/D port receives an analog voltage signal from the breadboard in a specified range (i.e., continuous voltages from 0 volts to 5 volts). The A/D port samples the analog signal at 1000 Hz (i.e., at 1 ms intervals). The 10 bit A/D converter on the microprocessor changes the 0 to 5 volt analog signal to bits (ones and zeros) that represent $2^{10} = 1024$ integer values (e.g., 5 volts converts to 1024 and 2.5 volts converts to 512). The Arduino transmits this data to the computer via serial communication.

- **Computer:**
  The computer receives bits from its serial port and uses the Arduino software program to translate the bits to integer numbers which are then printed to the screen. The numbers displayed on the computer screen are integers from 0 to 1024.

4.3.4 Determining the motor’s $\frac{\Omega}{v_{\text{motor}}}$ proportionality constant

The maximum (peak) of $|x_{ss}|$ versus $\Omega$ should occur at $\Omega \approx \omega_n$ (you previously determined $\omega_n$). Determine the linear-proportional constant\(^\text{11}\) that relates motor angular speed $\Omega$ to motor input voltage $v_{\text{motor}}$.

\[
\Omega \left( \frac{\text{rad}}{\text{sec}} \right) = \, * \, v_{\text{motor}} \left( \text{rad/s} \right)
\]

To complete the previous equation, use an oscilloscope to measure the value of $v_{\text{motor}}$ that corresponds to the maximum (peak) value of $|x_{ss}|$ before breaking out of frequency locking.

What is frequency locking? (Hint: Ask a TA).

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\(^9\)Most accelerometers do not come assembled with a microprocessor, transceiver, and computer.

\(^10\)NIST (National Institute of Standards and Technology) defines 1 $g$ as exactly 9.80665 m/s².

\(^11\)This assumes a directly proportional relationship between $\Omega$ and $v_{\text{motor}}$. 
4.3.5 Data acquisition

1. **Before** data acquisition, complete the $\Omega$ and $v_{\text{motor}}$ columns of the table below.

2. If necessary, login to the lab computer. Username: me161student  
   Password: 1euler1. Ensure the domain is ENGR

3. Power the Arduino by plugging-in (in order):
   (a). 12 Volt adaptor (between the board and wall socket)
   (b). USB cable (between the board and the computer)

4. From the desktop, navigate to the Lab2 folder and open Lab2.ino

5. Under Tools $\rightarrow$ ports, select something other than COM1, COM2, or COM3  
   (the USB port can be enumerated to anything other than these)

6. Click the magnifying glass button (or type Ctrl+Shift+m) to open the serial monitor

7. On the serial monitor screen, a menu should appear. Enter “a” to start reporting data.

8. Have one group member hold the cart

9. Input a desired PWM value (0-255) to select motor voltage for driving the cart.  
   **Do not select a voltage above 4 Volts (85 in the PWM selection).**

10. Verify the selected PWM value gives the desired voltage on the scope. If it matches, remove scope probes and proceed. If not, enter ‘r’ to reset, and adjust the PWM accordingly.

11. Release the cart.  
    Note: At low voltages, you may have to nudge the offset mass to overcome static friction and get the motor rotating.

12. Allow the cart to reach steady state, and press 'q' to get the max accelerometer reading at steady-state. Record this number in the table below.

13. Press 'r' to stop the motor and reset.

14. Convert the value from step 12 to volts.  
    Record this value under $|v_{\text{accelerometer}}|$. Using $v_{0g}$, convert $|v_{\text{accelerometer}}|$ to $|\ddot{x}_{ss}|$.

15. Repeat steps 8 through 14 to record $|\ddot{x}_{ss}|$ for $0 \leq \frac{\Omega}{\omega_n} \leq 2$ in the table below.

16. Ensure the power to the board is off and the setup is neat for the next lab.

| Approx. $\frac{\Omega}{\omega_n}$ | $\Omega$ (rad/sec) | $v_{\text{motor}}$ (volts) | PWM integer | Sensor | $v_{\text{accelerometer}}$ (volts) | $|\ddot{x}_{ss}|$ (m/s$^2$) |
|---|---|---|---|---|---|---|
| 0.50 | | | |
| 0.70 | | | |
| 0.80 | | | |
| 0.90 | | | |
| 1.00 | | | |
| 1.10 | | | |
| 1.20 | | | |
| 1.30 | | | |
| 1.50 | | | |
| 1.70 | | | |
| 1.90 | | | |

Plot your experimental data for $|\ddot{x}_{ss}|$ on your previous graph of analytical $|\ddot{x}_{ss}|$ versus $\Omega$.  
How well does the analytic model compare to the experimental data?  
(Circle one)  
Great $< 0.1\%$  
Very good $< 1\%$  
Good $< 10\%$  
Fair $< 50\%$  
Poor $< 100\%$

Identify potential sources of experimental errors.

- Errors in the physical plant include:
- Errors in the mechatronics include:
- Other errors include: