Chapter 2

Vectors and dyadics

Summary

Circa 1900 A.D., J. Williard Gibbs proposed the idea of vectors and their higher-dimensional counterparts dyadics, triadics, and polyadics. Vectors describe three-dimensional space and are an important geometrical tool for scientific and engineering fields, e.g., surveying, motion analysis, lasers, optics, computer graphics, animation, and CAD/CAE (computer aided drawing/engineering).

Vector and dyadic operations used in kinematic, static, and dynamic analysis include:

- Multiplication of a vector with a scalar (produces a vector)
- Multiplication of a vector with a vector (produces a dyadic)
- Vector addition and dyadic addition
- Dot product or cross product of a vector with a vector
- Dot product of a vector with a dyadic
- Differentiation of a vector

This chapter describes vectors and vector operations in a basis-independent way. Although it can be helpful to use an “x, y, z” or “i, j, k” orthogonal basis to represent vectors, it is not always necessary or desirable. Postponing the resolution of a vector into components is often computationally efficient, allowing for maximum use of basis-independent vector identities and avoids the necessity of simplifying trigonometric identities such as \( \sin^2(\theta) + \cos^2(\theta) = 1 \) (see Homework 2.8).

2.1 Examples of scalars, vectors, and dyadics

- A scalar is a quantity, e.g., a positive or negative number, that does not have an associated direction. For example, time, temperature, and density are scalar quantities.
- A vector is a quantity that has magnitude and one associated direction. For example, a velocity vector is a useful encapsulation of speed (how fast something is moving) with direction (which way it is going). A force vector is a succinct representation of its magnitude (how hard something is being pushed) with its direction (which way it is being shoved).
- A dyad is a quantity that has magnitude and two associated directions. For example, product of inertia is a measure of how far mass is distributed in two directions. Stress is associated with forces and areas (both regarded as vectors).
- A dyadic is the sum of dyads. For example, an inertia dyadic describes the mass distribution of a body and is the sum of various dyads associated with products and moments of inertia.

The following table lists a variety of quantities. Each quantity is identified as a scalar (no associated direction), vector (one associated direction), or dyad/dyadic (two associated directions).

<table>
<thead>
<tr>
<th>Scalar quantities</th>
<th>Vector quantities</th>
<th>Dyadic quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass</td>
<td>position vector</td>
<td>inertia dyadic</td>
</tr>
<tr>
<td>volume</td>
<td>velocity</td>
<td>stress dyadic</td>
</tr>
<tr>
<td>angle</td>
<td>acceleration</td>
<td>strain dyadic</td>
</tr>
<tr>
<td>moment of inertia</td>
<td>force</td>
<td></td>
</tr>
<tr>
<td>kinetic energy</td>
<td>torque</td>
<td></td>
</tr>
</tbody>
</table>
2.2 Definition of a vector

A vector is defined as a quantity having magnitude and direction. Vectors are represented graphically with straight or curved arrows. For example, the vectors depicted below are directed to the right, left, up, down, out from the page, into the page, and inclined at 45°, respectively.

Certain vectors have special properties (in addition to magnitude and direction) and have special names to reflect these additional properties. For example, a position vector is associated with two points and has units of distance. A bound vector is associated with a point (or a line of action).

Example of a vector

Traffic reports include observations such as “the vehicle is heading East at 5 m/sec”. In engineering, it is conventional to represent these two pieces of information, namely the vehicle’s speed (5 m/sec) and its direction (East) by putting them next to each other or multiplying them (5 East). To clearly distinguish the speed from the direction, it is common to put an arrow over the direction (\(\vec{\text{East}}\)), or to use bold-face font (East), or to use a hat for a unit vector (\(\hat{\text{East}}\)).

The vehicle’s speed is always a non-negative number. Generically, this non-negative number is called the magnitude of the vector. The combination of magnitude and direction is a vector.

For example, the vector \(\mathbf{v}\) describing a vehicle traveling with speed 5 to the East is graphically depicted to the right, and is written

\[ \mathbf{v} = 5 \text{ East} \quad \text{or} \quad \mathbf{v} = 5 \text{East} \]

A vehicle traveling with speed 5 to the West is

\[ 5 \text{ West} \quad \text{or} \quad -5 \text{ East} \]

Note: The negative sign in “-5 East” is associated with the vector’s direction (the vector’s magnitude is inherently positive).

2.3 The zero vector 0

The zero vector \(\mathbf{0}\) is defined as a vector whose magnitude is zero.

The zero vector may have any direction and has the following properties.

- Addition of a vector \(\mathbf{a}\) with the zero vector: \(\mathbf{a} + \mathbf{0} = \mathbf{a}\)
- Dot product with the zero vector: \(\mathbf{a} \cdot \mathbf{0} = 0\)
- Cross product with the zero vector: \(\mathbf{a} \times \mathbf{0} = \mathbf{0}\)

Note: Direction can be resolved into orientation and sense. For example, a highway has an orientation (e.g., east-west) and a vehicle traveling east has a sense. Knowing both the orientation of a line and the sense on the line gives direction.

In Autolev, > is used to denote a vector, e.g., the vector \(\mathbf{v}\) is represented as \(\mathbf{v}\) and the zero vector is \(\mathbf{0}\).

Note: It is improper to say the zero vector has no direction as a vector is defined to have both a magnitude and a direction. It is also improper to say the zero vector has all directions as a vector is defined to have a magnitude and a direction (as contrasted with a dyad which has two directions, triad which has three directions, etc.). Thus there are an infinite number of equal zero vectors, each having zero magnitude and any direction.
2.4 Unit vectors

A unit vector is defined as a vector whose magnitude is 1. Unit vectors are sometimes designated with a special vector hat, e.g., \( \hat{u} \).

Unit vectors are typically introduced as “sign posts”, e.g., the unit vectors North, South, West, and East shown to the right. The direction of unit vectors are chosen to simplify communication and to produce efficient equations. Other useful “sign posts” are:

- unit vector directed from one point to another point
- unit vector directed locally vertical
- unit vector tangent to a curve
- unit vector parallel to the edge of an object
- unit vector perpendicular to a surface

Another way to introduce a unit vector is to define it so it has the same direction as an arbitrary non-zero vector \( \mathbf{v} \) by first determining \( |\mathbf{v}| \), the magnitude of \( \mathbf{v} \), and then writing

\[
\text{unitVector} = \frac{\mathbf{v}}{|\mathbf{v}| + \epsilon}
\]

\(^a\)Homework 2.4 draws various unit vectors. \(^b\)To avoid divide-by-zero problems during numerical computation, one may write the unit vector in terms of a “small” positive constant \( \epsilon \) as

\[
\text{unitVector} = \frac{\mathbf{v}}{|\mathbf{v}| + \epsilon}
\]

2.5 Equal vectors

Two vectors are said to be equal (or equivalent) when they have the same magnitude and same direction.\(^a\) The figure to the right shows three equal vectors. Although each vector has a different location, the vectors are equal because they have the same magnitude and direction.\(^b\)

\(^a\)Homework 2.3 draws vectors of different magnitude, orientation, and sense. \(^b\)Certain vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are equal position vectors when, in addition to having the same magnitude and direction, the vectors are associated with the same points. Two force vectors are equal force vectors when the vectors have the same magnitude, direction, and point of application.

2.6 Vector addition

As graphically shown to the right, adding two vectors \( \mathbf{a} + \mathbf{b} \) produces a vector.\(^a\) First, vector \( \mathbf{b} \) is translated\(^b\) so its tail is at the tip of \( \mathbf{a} \). Next, the vector \( \mathbf{a} + \mathbf{b} \) is drawn from the tail of \( \mathbf{a} \) to the tip of the translated \( \mathbf{b} \).\(^c\)

**Properties of vector addition**

- Commutative law: \( \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \)
- Associative law: \( (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = \mathbf{a} + \mathbf{b} + \mathbf{c} \)
- Addition of zero vector: \( \mathbf{a} + 0 = \mathbf{a} \)

\(^a\)It does not make sense to add vectors with different units. For example, adding a velocity vector with units of m/sec with an angular velocity vector with units of rad/sec does not produce a vector with sensible units. \(^b\)Translating \( \mathbf{b} \) does not change the magnitude or direction of \( \mathbf{b} \), and so produces an equal \( \mathbf{b} \). \(^c\)Homework 2.7 draws \( \mathbf{b} + \mathbf{a} \).
2.7 Vector negation

A graphical representation of negating a vector \( \mathbf{a} \) is shown to the right.\(^a\)

Negating a vector (multiplying the vector by -1) changes the sense of a vector without changing its magnitude or orientation. In other words, multiplying a vector by -1 reverses the sense of the vector (it points in the opposite direction).

\(^a\)Homework 2.5 draws a vector \(-\mathbf{b}\).

2.8 Vector multiplied or divided by a scalar

To the right is a graphical representation of multiplying a vector \( \mathbf{a} \) by a scalar.\(^a\)

- Multiplying a vector by a positive number (other than 1) changes the vector’s magnitude.
- Multiplying a vector by a negative number changes the vector’s magnitude and reverses the sense of the vector.
- Dividing a vector \( \mathbf{a} \) by a scalar \( s_1 \) is defined as \( \frac{\mathbf{a}}{s_1} \triangleq \frac{1}{s_1} \ast \mathbf{a} \)

Properties of multiplication of a vector by a scalar

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative law</td>
<td>( s_1 \mathbf{a} = \mathbf{a} s_1 )</td>
</tr>
<tr>
<td>Associative law</td>
<td>( s_1 (s_2 \mathbf{a}) = (s_1 s_2) \mathbf{a} = s_2 (s_1 \mathbf{a}) = s_1 s_2 \mathbf{a} )</td>
</tr>
<tr>
<td>Distributive law</td>
<td>( (s_1 + s_2) \mathbf{a} = s_1 \mathbf{a} + s_2 \mathbf{a} )</td>
</tr>
<tr>
<td>Distributive law</td>
<td>( s_1 (\mathbf{a} + \mathbf{b}) = s_1 \mathbf{a} + s_1 \mathbf{b} )</td>
</tr>
<tr>
<td>Multiplication by one</td>
<td>( 1 \ast \mathbf{a} = \mathbf{a} )</td>
</tr>
<tr>
<td>Multiplication by zero</td>
<td>( 0 \ast \mathbf{a} = \mathbf{0} )</td>
</tr>
</tbody>
</table>

\(^a\)Homework 2.6 multiplies a vector \( \mathbf{b} \) by various scalars.

2.9 Vector subtraction

As graphically shown to the right, the process of subtracting a vector \( \mathbf{b} \) from a vector \( \mathbf{a} \) is simply addition and negation,\(^a\) i.e.,

\[ \mathbf{a} - \mathbf{b} \triangleq \mathbf{a} + -\mathbf{b} \]

After negating vector \( \mathbf{b} \), it is translated so the tail of \(-\mathbf{b}\) is at the tip of \( \mathbf{a} \). Next, the vector \( \mathbf{a} + -\mathbf{b} \) is drawn from the tail of \( \mathbf{a} \) to the tip of the translated \(-\mathbf{b}\).\(^b\)

\(^a\)In most (or all) mathematical processes, subtraction is defined as negation and addition.

\(^b\)Homework 2.8 draws \( \mathbf{b} - \mathbf{a} \triangleq \mathbf{b} + -\mathbf{a} \).
2.10 Vector dot product

The **dot product** of a vector \( \mathbf{a} \) with a vector \( \mathbf{b} \) is defined in equation (2) in terms of
- \( |\mathbf{a}| \) and \( |\mathbf{b}| \), the magnitudes of \( \mathbf{a} \) and \( \mathbf{b} \), respectively
- \( \theta \), the smallest angle between \( \mathbf{a} \) and \( \mathbf{b} \) (\( 0 \leq \theta \leq \pi \)).

When \( \mathbf{b} \) is a unit vector, \( |\mathbf{b}|=1 \) and \( \mathbf{a} \cdot \mathbf{b} \) can be interpreted as the projection of \( \mathbf{a} \) on \( \mathbf{b} \). Similarly, when \( \mathbf{a} \) is a unit vector, \( \mathbf{a} \cdot \mathbf{b} \) can be interpreted as the projection of \( \mathbf{b} \) on \( \mathbf{a} \).

Rearranging equation (2) produces an expression which is useful for finding the angle between two vectors, i.e., \( \theta = \arccos \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \right) \)

![Vector dot product diagram](image)

The dot product is useful for calculating the magnitude of a vector \( \mathbf{v} \). In view of equation (2), \( \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \). Hence, one way to determine \( |\mathbf{v}| \) is the important relationship in equation (3).

2.10.1 Properties of the dot-product

- **Dot product with the zero vector** \( \mathbf{a} \cdot \mathbf{0} = 0 \)
- **Dot product of perpendicular vectors** \( \mathbf{a} \cdot \mathbf{b} = 0 \) if \( \mathbf{a} \perp \mathbf{b} \)
- **Dot product of vectors having the same direction** \( \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \) if \( \mathbf{a} \parallel \mathbf{b} \)
- **Dot product with vectors scaled by \( s_1 \) and \( s_2 \)** \( s_1 \mathbf{a} \cdot s_2 \mathbf{b} = s_1 s_2 (\mathbf{a} \cdot \mathbf{b}) \)
- **Commutative law** \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \)
- **Distributive law** \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)
- **Distributive law** \( (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d} \)

2.10.2 Uses for the dot-product

Several uses for the dot-product in geometry, statics, and motion analysis, include
- Calculating an **angle** between two vectors (very useful in geometry)
- Calculating a vector’s **magnitude** (e.g., **distance** is the magnitude of a position vector)
- Calculating a **unit vector** in the direction of a vector [as shown in equation (1)]
- Determining when two vectors are **perpendicular**
- Determining the **component** (or measure) of a vector in a certain direction
- Changing a **vector equation** into a **scalar equation** (see Homework 2.19)
- **Vector exponentiation**
  
  The definition of **vector exponentiation** of \( \mathbf{v}^n \) for the vector \( \mathbf{v} \) raised to the scalar power \( n \) and the specific case of \( \mathbf{v}^2 \) are shown to the right. Note: \( \mathbf{v}^n \) produces a **non-negative** scalar.

\[
\begin{align*}
\mathbf{v}^n & \triangleq |\mathbf{v}|^n = +(|\mathbf{v} \cdot \mathbf{v}|)^{\frac{n}{2}} \\
\mathbf{v}^2 & \triangleq |\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} 
\end{align*}
\]

2.10.3 Special case: Dot-products with orthogonal unit vectors

When \( \mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z \) are **orthogonal unit** vectors, it can be shown (see Homework 2.6)

\[
(a_x \mathbf{n}_x + a_y \mathbf{n}_y + a_z \mathbf{n}_z) \cdot (b_x \mathbf{n}_x + b_y \mathbf{n}_y + b_z \mathbf{n}_z) = a_x b_x + a_y b_y + a_z b_z
\]
2.10.4 Example: Microphone cable lengths and angles (with orthogonal walls)

A microphone $Q$ is attached to three pegs $A$, $B$, and $C$ by three cables. Knowing the peg locations and microphone location from point $N_o$, determine $L_A$ (the length of the cable joining $A$ and $Q$) and the angle $\phi$ between line $AQ$ and line $AB$.

![Diagram of microphone and pegs]

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance from $A$ to $B$</td>
<td>20 m</td>
</tr>
<tr>
<td>Distance from $B$ to $C$</td>
<td>15 m</td>
</tr>
<tr>
<td>Distance from $N_o$ to $B$</td>
<td>8 m</td>
</tr>
<tr>
<td>$Q$'s measure from $N_o$ along back-wall</td>
<td>7 m</td>
</tr>
<tr>
<td>$Q$'s height above $N_o$</td>
<td>5 m</td>
</tr>
<tr>
<td>$Q$'s measure from $N_o$ along left-wall</td>
<td>8 m</td>
</tr>
</tbody>
</table>

\[ r^{Q/N_o} = 7n_x + 5n_y + 8n_z \]

**Solution:**

- Form $A$’s position vector from $N_o$ (inspection): \[ r^{A/N_o} = 8n_y + 20n_z \]
- Form $Q$’s position vector from $A$ (vector addition and rearrangement):
  \[ r^{Q/A} = r^{Q/N_o} - r^{A/N_o} = 7n_x -3n_y -12n_z \]
- Calculate \( r^{Q/A} \cdot r^{Q/A} \): \[ (7n_x -3n_y -12n_z) \cdot (7n_x +3n_y +12n_z) = 202 \]
- Form $L_A$, the magnitude of $Q$’s position vector from $A$:
  \[ L_A = \sqrt{r^{Q/A} \cdot r^{Q/A}} = \sqrt{202} = 14.2 \]
- The determination of the angle $\phi$ starts with the definition of the following dot-product
  \[ r^{Q/A} \cdot r^{B/A} = \frac{|r^{Q/A}| |r^{B/A}| \cos(\phi)}{20} \]
- Subsequent rearrangement and substitution of the known quantities gives
  \[ \cos(\phi) = \frac{r^{Q/A} \cdot r^{B/A}}{|r^{Q/A}| |r^{B/A}|} = \frac{20L_A}{(7n_x -3n_y -12n_z) \cdot (-20n_z)} = \frac{240}{284} \]
- Solving for the angle gives $\phi = \text{acos}(\frac{240}{284}) = 32.32^\circ$. 

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2.11 Vector cross product

The **cross product** of a vector \( \mathbf{a} \) with a vector \( \mathbf{b} \) is defined in equation (5) in terms of

- \( |\mathbf{a}| \) and \( |\mathbf{b}| \), the magnitudes of \( \mathbf{a} \) and \( \mathbf{b} \), respectively
- \( \theta \), the smallest angle between \( \mathbf{a} \) and \( \mathbf{b} \) \( (0 \leq \theta \leq \pi) \).
- \( \mathbf{u} \), the unit vector perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \) whose direction is determined by the **right-hand rule**.\(^a\)

Note: The coefficient of \( \mathbf{u} \) in equation (5) is inherently a **non-negative** quantity since \( \sin(\theta) \geq 0 \) because \( 0 \leq \theta \leq \pi \). Hence, \( \mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{u} \) \( (5) \)

\(^a\)The right-hand rule is a recently accepted universal convention, much like driving on the right-hand side of the road in North America. Until 1965, the Soviet Union used the left-hand rule, logically reasoning that the left-hand rule is more convenient because a right-handed person can simultaneously write while performing cross products.

2.11.1 Properties of the cross-product

Cross product with the zero vector  
\[ \mathbf{a} \times \mathbf{0} = \mathbf{0} \]

Cross product of a vector with itself  
\[ \mathbf{a} \times \mathbf{a} = \mathbf{0} \]

Cross product of parallel vectors  
\[ \mathbf{a} \times \mathbf{b} = \mathbf{0} \quad \text{if} \quad \mathbf{a} \parallel \mathbf{b} \]

Cross product with vectors scaled by \( s_1 \) and \( s_2 \)  
\[ s_1 \mathbf{a} \times s_2 \mathbf{b} = s_1 s_2 (\mathbf{a} \times \mathbf{b}) \]

Cross products are **not** commutative  
\[ \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \]

Distributive law  
\[ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \]

Distributive law  
\[ (\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d} \]

Cross products are not associative  
\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \]

**Vector triple cross product**  
When \( \mathbf{b} \) is a unit vector,  
\[ |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \]

A mnemonic for \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \) is **back cab** - as in were you born in the back of a cab?

Many proofs of this formula resolve \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) into orthogonal unit vectors (e.g., \( \mathbf{n}_x \), \( \mathbf{n}_y \), \( \mathbf{n}_z \)) and equate components.

2.11.2 Uses for the cross-product

Several uses for the cross-product in geometry, statics, and motion analysis, include

- Calculating **perpendicular** vectors, e.g., \( \mathbf{v} = \mathbf{a} \times \mathbf{b} \) is perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \)
- Determining when two vectors are **parallel**, e.g., \( \mathbf{a} \times \mathbf{b} = \mathbf{0} \) when \( \mathbf{a} \) is parallel to \( \mathbf{b} \)
- Calculating the **moment** of a force or linear momentum, e.g., \( \mathbf{M} = \mathbf{r} \times \mathbf{F} \) and \( \mathbf{H} = \mathbf{r} \times m \mathbf{v} \)
- Calculating **velocity/acceleration** formulas, e.g., \( \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \) and \( \mathbf{a} = \mathbf{\alpha} \times \mathbf{r} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \)
- Calculating the **area of a triangle** whose sides have length \( |\mathbf{a}| \) and \( |\mathbf{b}| \)

The **area of a triangle** \( \Delta \) is half the area of a parallelogram.\(^a\) Since one geometrical interpretation of \( |\mathbf{a} \times \mathbf{b}| \) is the **area of a parallelogram** having sides of length \( |\mathbf{a}| \) and \( |\mathbf{b}| \),

\[ \Delta = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| \]  \( (6) \)

\(^a\)Homework 2.11 shows the utility of equation (6) for **surveying**.
2.11.3 Special case: Cross-products with right-handed, orthogonal, unit vectors

Given right-handed orthogonal unit vectors \( \mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z \) and two arbitrary vectors \( \mathbf{a} \) and \( \mathbf{b} \) that are expressed in terms of \( \mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z \) as shown to the right, calculating \( \mathbf{a} \times \mathbf{b} \) with the distributive property of the cross product happens to be equal to the determinant of the matrix and the expression shown below.\(^a\)

\[
\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = (a_y b_z - a_z b_y) \mathbf{n}_x + (a_z b_x - a_x b_z) \mathbf{n}_y + (a_x b_y - a_y b_x) \mathbf{n}_z
\]

2.12 Scalar triple product and the volume of a tetrahedron

The scalar triple product of vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) is the scalar defined in the various ways shown in equation (7).\(^4\) Homework 2.13 shows how determinants can calculate certain scalar triple products.

\[
\text{ScalarTripleProduct} \triangleq \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} \quad (7)
\]

A geometrical interpretation of \( \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \) is the volume of a parallelepiped having sides of length \( |\mathbf{a}|, |\mathbf{b}|, \) and \( |\mathbf{c}| \). The formula for the volume of a tetrahedron whose sides are described by the vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \)

\[
\text{Tetrahedron Volume} = \frac{1}{6} \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}
\]

This formula is useful for volume (surveying cut and fill) calculations as well as 3D (CAD) solid geometry mass property calculations.

2.13 Dyads and dyadics (vector multiplied by a vector)

The dyad \( \mathbf{D} \) that results from the multiplication of the vectors \( \mathbf{a} \) and \( \mathbf{b} \) is defined as \( \mathbf{D} \triangleq \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \mathbf{b} \). To clearly distinguish a dyad from a scalar or vector, it is common to use two arrows, two hats, two tildes, or two underlines, i.e., \( \mathbf{D}, \hat{\mathbf{D}}, \tilde{\mathbf{D}}, \) or \( \mathbf{D} \), or to use a single line under a bold-face symbol, i.e., \( \mathbf{D} \).\(^5\)

The sum of two or more dyads is called a dyadic. For example, letting \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \) and \( \mathbf{d} \) be vectors, the dyads \( \mathbf{a} \mathbf{b} \) and \( \mathbf{c} \mathbf{d} \) can be formed. Their sum \( \mathbf{a} \mathbf{b} + \mathbf{c} \mathbf{d} \) is a dyadic.\(^6\)

2.13.1 The zero dyadic and the unit dyadic

The zero dyadic \( \mathbf{0} \) is defined as the dyad with two zero vectors.

\[
\mathbf{0} \triangleq 0 \times 0 \quad (8)
\]

\(^a\)This is proved in Homework 2.9.

\(^4\)Although parentheses make equation (7) clearer, i.e., \( \text{ScalarTripleProduct} \triangleq \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \), the parentheses are unnecessary because the cross product \( \mathbf{b} \times \mathbf{c} \) must be performed before the dot product for a sensible result to be produced.

\(^5\)In AutoLev, \( >> \) is used to denote a dyadic, e.g., the dyadic \( \mathbf{D} \) is represented as \( \mathbf{D} >> \) and the unit dyadic is \( \mathbf{1} >> \).

\(^6\)In general, the dyadic \( \mathbf{a} \mathbf{b} + \mathbf{c} \mathbf{d} \) is not a dyad because it cannot be written as a vector multiplied by a vector. Dyadics differ from vectors in that the sum of two vectors is a vector whereas the sum of two dyads is not necessarily a dyad.
The **unit dyadic** is denoted \( \mathbf{1} \) and is defined by its property in equation (9).\(^a\) As shown in Section 2.14, the unit dyadic can be written in terms of the orthogonal unit vectors \( \mathbf{b}_x, \mathbf{b}_y, \mathbf{b}_z \) as shown in equation (10).\(^b\)

\[^a\text{The unit dyadic is similar to the identity matrix } I \text{ whose defining property is}\]
\[^b\text{The orthogonal unit vectors } \mathbf{b}_x, \mathbf{b}_y, \mathbf{b}_z \text{ do not have to be right-handed for the unit dyadic to be expressed as } \mathbf{1} = \mathbf{b}_x \mathbf{b}_x + \mathbf{b}_y \mathbf{b}_y + \mathbf{b}_z \mathbf{b}_z. \text{ Unless a cross-product is involved, the right-handed nature of the vectors is irrelevant.}\]

\[\mathbf{1} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{1} = \mathbf{v}\]

where \( \mathbf{v} \) is any vector.

\[\mathbf{1} = \mathbf{b}_x \mathbf{b}_x + \mathbf{b}_y \mathbf{b}_y + \mathbf{b}_z \mathbf{b}_z\]  

(10)

### 2.13.2 Properties of dyadics associated with the vectors \( a, b, c, d, \) and \( w \)

Dyads are not commutative: \( ab \neq ba \)

- **Distributive law:** \( a (b + c) = ab + ac \)
- **Distributive law:** \( (a + b) (c + d) = ac + ad + bc + bd \)
- **Pre-dot product:** \( \mathbf{w} \cdot (\mathbf{a} + \mathbf{c}) = (\mathbf{w} \cdot \mathbf{a}) \mathbf{b} + (\mathbf{w} \cdot \mathbf{c}) \mathbf{d} \)
- **Post-dot product:** \( (\mathbf{a} + \mathbf{c}) \cdot \mathbf{w} = \mathbf{a} (\mathbf{b} \cdot \mathbf{w}) + \mathbf{c} (\mathbf{d} \cdot \mathbf{w}) \)
- **Pre-cross product:** \( \mathbf{w} \times (\mathbf{a} + \mathbf{c}) = (\mathbf{w} \times \mathbf{a}) \mathbf{b} + (\mathbf{w} \times \mathbf{c}) \mathbf{d} \)
- **Post-cross product:** \( (\mathbf{a} + \mathbf{c}) \times \mathbf{w} = \mathbf{a} (\mathbf{b} \times \mathbf{w}) + \mathbf{c} (\mathbf{d} \times \mathbf{w}) \)
- **Vector multiplication:** \( s_1 \mathbf{a} \times s_2 \mathbf{b} = s_1 s_2 \mathbf{a} \mathbf{b} \) \((s_1 \text{ and } s_2 \text{ are scalars})\)
- **Dot product with } \mathbf{0}: \) \( \mathbf{0} \cdot \mathbf{0} = 0 \) \((\mathbf{0} \text{ is any dyadic and } \mathbf{0} \text{ is the zero vector})\)

### 2.13.3 Dyadic examples

Complete the following calculations using the orthogonal unit vectors \( \mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z. \)

\[
\begin{align*}
\mathbf{0} \cdot (\mathbf{a}_x + 5 \mathbf{a}_y) &= \\
\mathbf{1} \cdot (\mathbf{a}_x + 5 \mathbf{a}_y) &= \\
\mathbf{1} &= \mathbf{a}_x \mathbf{a}_x + \mathbf{a}_y \mathbf{a}_y + \\
(\mathbf{a}_x + 2 \mathbf{a}_y + 3 \mathbf{a}_z) \cdot (4 \mathbf{a}_x + 5 \mathbf{a}_y + 6 \mathbf{a}_z) &= 4 \mathbf{a}_x \mathbf{a}_x + 5 \mathbf{a}_x \mathbf{a}_y + \\
&+ \mathbf{a}_x \mathbf{a}_z + \mathbf{a}_y \mathbf{a}_z + \mathbf{a}_z \mathbf{a}_z + 18 \mathbf{a}_z \mathbf{a}_z \\
(\mathbf{a}_y \mathbf{a}_z + 4 \mathbf{a}_x \mathbf{a}_z + 5 \mathbf{a}_z \mathbf{a}_y) \cdot (\mathbf{a}_x + 2 \mathbf{a}_y + 3 \mathbf{a}_z) &= 3 \mathbf{a}_y + \\
(\mathbf{a}_x + 2 \mathbf{a}_y + 3 \mathbf{a}_z) \cdot (\mathbf{a}_y \mathbf{a}_z + 4 \mathbf{a}_x \mathbf{a}_z + 5 \mathbf{a}_z \mathbf{a}_y) &= 2 \mathbf{a}_z +
\end{align*}
\]

Complete the next set of calculations with the orthogonal unit vectors \( \mathbf{b}_x, \mathbf{b}_y, \mathbf{b}_z. \)

\[
\begin{align*}
(\mathbf{a}_x + 2 \mathbf{a}_y) \times (\mathbf{b}_x + 3 \mathbf{b}_z) &= \mathbf{a}_x \mathbf{b}_x + 3 \mathbf{a}_x \mathbf{b}_z + 2 \mathbf{a}_y \mathbf{b}_x +
\\
(\mathbf{a}_x \mathbf{b}_x + 3 \mathbf{a}_x \mathbf{b}_z + 2 \mathbf{a}_y \mathbf{b}_x + 6 \mathbf{a}_y \mathbf{b}_z) \cdot \mathbf{b}_x &= \mathbf{a}_x +
\\
\mathbf{a}_y \cdot (\mathbf{a}_x \mathbf{b}_x + 3 \mathbf{a}_x \mathbf{b}_z + 2 \mathbf{a}_y \mathbf{b}_x + 6 \mathbf{a}_y \mathbf{b}_z) &= 2 \mathbf{b}_x +
\end{align*}
\]

### 2.14 Optional**: Unit dyadic expressed with orthogonal unit vectors

To prove \( \mathbf{1} = \mathbf{b}_x \mathbf{b}_x + \mathbf{b}_y \mathbf{b}_y + \mathbf{b}_z \mathbf{b}_z \) in equation (10), note that equations (3.1) and (3.4) allow an arbitrary vector \( \mathbf{v} \) to be expressed in terms of any orthogonal unit vectors \( \mathbf{b}_x, \mathbf{b}_y, \mathbf{b}_z \) as

\[
\mathbf{v} = (\mathbf{v} \cdot \mathbf{b}_x) \mathbf{b}_x + (\mathbf{v} \cdot \mathbf{b}_y) \mathbf{b}_y + (\mathbf{v} \cdot \mathbf{b}_z) \mathbf{b}_z = \mathbf{v} \cdot (\mathbf{b}_x \mathbf{b}_x + \mathbf{b}_y \mathbf{b}_y + \mathbf{b}_z \mathbf{b}_z)
\]

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As described in Section 2.13, the unit dyadic is defined by its property \( \mathbf{v} \cdot \mathbf{1} = \mathbf{v} \), hence

\[
\mathbf{v} \cdot \mathbf{1} = \mathbf{v} \cdot (b_x \mathbf{b}_x + b_y \mathbf{b}_y + b_z \mathbf{b}_z)
\]

or

\[
\mathbf{v} \cdot [\mathbf{1} - (b_x \mathbf{b}_x + b_y \mathbf{b}_y + b_z \mathbf{b}_z)] = 0
\]

The proof is completed by noting that \( \mathbf{v} \) is an arbitrary vector (e.g., not-necessarily 0).

### 2.15 Vector operations with Autolev

Vector operations such as addition, scalar multiplication, dot-products, and cross-products can be performed with Autolev as shown below.

1. % File: VectorDemonstration.al % The percent sign denotes a comment
2. RigidFrame A % Create orthogonal unit vectors \( \mathbf{Ax}, \mathbf{Ay}, \mathbf{Az} \)
3. \( \mathbf{V} = \text{Vector}( A, 2, 3, 4 ) \) % Construct a vector \( \mathbf{V} \)
   - \( \mathbf{V} = 2 \mathbf{Ax} + 3 \mathbf{Ay} + 4 \mathbf{Az} \)
4. \( \mathbf{W} = \text{Vector}( A, 6, 7, 8 ) \) % Construct a vector \( \mathbf{W} \)
   - \( \mathbf{W} = 6 \mathbf{Ax} + 7 \mathbf{Ay} + 8 \mathbf{Az} \)
5. \( \mathbf{V5} = 5 \mathbf{V} \) % Multiply \( \mathbf{V} \) by 5
   - \( \mathbf{V5} = 10 \mathbf{Ax} + 15 \mathbf{Ay} + 20 \mathbf{Az} \)
6. \( \text{magV} = \text{GetMagnitude}( \mathbf{V} ) \) % Magnitude of \( \mathbf{V} \)
   - \( \text{magV} = 5.385165 \)
7. \( \text{unitV} = \text{GetUnitVector}( \mathbf{V} ) \) % Unit vector in the direction of \( \mathbf{V} \)
   - \( \text{unitV} = 0.3713907 \mathbf{Ax} + 0.557086 \mathbf{Ay} + 0.7427814 \mathbf{Az} \)
8. \( \text{addVW} = \mathbf{V} + \mathbf{W} \) % Add vectors \( \mathbf{V} \) and \( \mathbf{W} \)
   - \( \text{addVW} = 8 \mathbf{Ax} + 10 \mathbf{Ay} + 12 \mathbf{Az} \)
9. \( \text{dotVW} = \text{Dot}( \mathbf{V}, \mathbf{W} ) \) % Dot product of \( \mathbf{V} \) and \( \mathbf{W} \)
   - \( \text{dotVW} = 65 \)
10. \( \text{crossVW} = \text{Cross}( \mathbf{V}, \mathbf{W} ) \) % Cross product of \( \mathbf{V} \) and \( \mathbf{W} \)
    - \( \text{crossVW} = -4 \mathbf{Ax} + 8 \mathbf{Ay} - 4 \mathbf{Az} \)
11. \( \text{crossWWV} = \text{Cross}( \mathbf{W}, \text{Cross}(\mathbf{W},\mathbf{V}) ) \) % Vector triple cross product
    - \( \text{crossWWV} = 92 \mathbf{Ax} + 8 \mathbf{Ay} - 76 \mathbf{Az} \)
12. \( \text{multVW} = \mathbf{V} \ast \mathbf{W} \) % Form a dyadic by multiplying \( \mathbf{V} \) and \( \mathbf{W} \)
    - \( \text{multVW} = 12 \mathbf{Ax} \ast \mathbf{Ax} + 14 \mathbf{Ax} \ast \mathbf{Ay} + 16 \mathbf{Ax} \ast \mathbf{Az} + 18 \mathbf{Ay} \ast \mathbf{Ax} + 21 \mathbf{Ay} \ast \mathbf{Ay} + 24 \mathbf{Ay} \ast \mathbf{Az} + 28 \mathbf{Az} \ast \mathbf{Ax} + 32 \mathbf{Az} \ast \mathbf{Ay} + 32 \mathbf{Az} \ast \mathbf{Az} \)
13. \( \text{dotVWithZeroVector} = \text{Dot}( \mathbf{V}, \mathbf{O} ) \) % Dot product of \( \mathbf{V} \) with the zero vector
    - \( \text{dotVWithZeroVector} = 0 \)
14. \( \text{dotVWithUnitDyadic} = \text{Dot}( \mathbf{V}, \mathbf{1} ) \) % Dot product of \( \mathbf{V} \) with the unit dyadic
    - \( \text{dotVWithUnitDyadic} = 2 \mathbf{Ax} + 3 \mathbf{Ay} + 4 \mathbf{Az} \)