Chapter 2

Vectors and dyadics

Summary

Circa 1900 A.D., J. Williard Gibbs proposed the idea of ***vectors*** and their higher-dimensional counterparts ***dyadics***, ***triadics***, and ***polyadics***. Vectors describe three-dimensional space and are an important **geometrical tool** for scientific and engineering fields, e.g., surveying, motion analysis, lasers, optics, computer graphics, animation, and CAD/CAE (computer aided drawing/engineering).
Vector and dyadic operations used in kinematic, static, and dynamic analysis include:

- **Multiplication of a vector with a scalar** (produces a vector)
- **Multiplication of a vector with a vector** (produces a dyadic)
- **Vector addition and dyadic addition**
- **Dot product or cross product of a vector with a vector**
- **Dot product of a vector with a dyadic**
- **Differentiation of a vector**

This chapter describes vectors and vector operations in a basis-independent way. Although it can be helpful to use an "**x**, **y**, **z**" or "**i**, **j**, **k**" orthogonal basis to represent vectors, it is not always necessary or desirable. Postponing the resolution of a vector into components is often computationally efficient, allowing for maximum use of basis-independent vector identities and avoids the necessity of simplifying trigonometric identities such as $\sin^2(\theta) + \cos^2(\theta) = 1$ (see Homework 2.8).

2.1 Examples of scalars, vectors, and dyadics

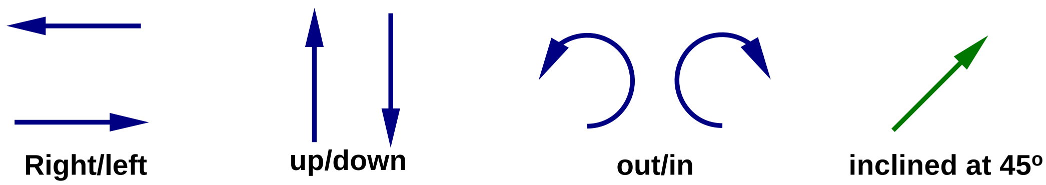
- A ***scalar*** is a quantity, e.g., a positive or negative number, that does *not* have an associated direction. For example, time, temperature, and density are scalar quantities.
- A ***vector*** is a quantity that has magnitude and ***one*** associated direction. For example, a ***velocity vector*** is a useful encapsulation of speed (how fast something is moving) with direction (which way it is going). A ***force vector*** is a succinct representation of its magnitude (how hard something is being pushed) with its direction (which way is it being shoved).
- A ***dyad*** is a quantity that has magnitude and ***two*** associated directions. For example, product of inertia is a measure of how far mass is distributed in two directions. Stress is associated with forces and areas (both regarded as vectors).
- A ***dyadic*** is the **sum** of dyads. For example, an ***inertia dyadic*** describes the mass distribution of a body and is the sum of various dyads associated with products and moments of inertia.

The following table lists a variety of quantities. Each quantity is identified as a scalar (no associated direction), vector (one associated direction), or dyad/dyadic (two associated directions).

Scalar quantities		**Vector quantities**		**Dyadic quantities**
mass	distance	position vector		inertia dyadic
volume	speed	velocity	angular velocity	stress dyadic
angle	potential energy	acceleration	angular acceleration	strain dyadic
moment of inertia	kinetic energy	force	torque	

2.2 Definition of a vector

A ***vector*** is defined as a quantity having ***magnitude*** and ***direction***.[1] Vectors are represented graphically with straight or curved arrows. For example, the vectors depicted below are directed to the right, left, up, down, out from the page, into the page, and inclined at 45°, respectively.

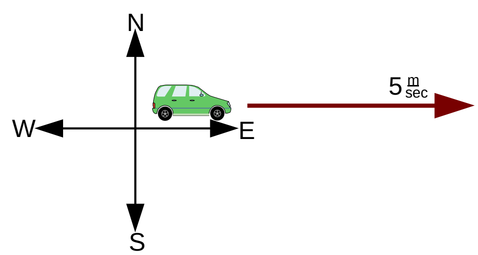


Certain vectors have special properties (in addition to magnitude and direction) and have special names to reflect these additional properties. For example, a ***position vector*** is associated with two points and has units of distance. A ***bound vector*** is associated with a point (or a line of action).

Example of a vector

Traffic reports include observations such as "the vehicle is heading East at $5\,\frac{\text{m}}{\text{sec}}$". In engineering, it is ***conventional*** to represent these two pieces of information, namely the vehicle's speed ($5\,\frac{\text{m}}{\text{sec}}$) and its direction (East) by putting them next to each other or multiplying them ($5 * \text{East}$). To clearly distinguish the speed from the direction, it is common to put an arrow over the direction ($\vec{\text{East}}$), or to use bold-face font (**East**), or to use a hat for a ***unit vector*** ($\widehat{\text{East}}$).[2] The vehicle's speed is always a non-negative number. Generically, this non-negative number is called the ***magnitude*** of the vector. The combination of ***magnitude*** and ***direction*** is a ***vector***.

For example, the vector **v** describing a vehicle traveling with speed 5 to the East is graphically depicted to the right, and is written



$$\mathbf{v} \;=\; 5 * \textbf{East} \qquad \text{or} \qquad \mathbf{v} \;=\; 5\,\textbf{East}$$

A vehicle traveling with speed 5 to the West is

$$5\,\textbf{West} \qquad \text{or} \qquad \text{-}5\,\textbf{East}$$

Note: The negative sign in -5 **East** is associated with the vector's direction (the vector's magnitude is inherently positive).

2.3 The zero vector 0

The ***zero vector*** **0** is defined as a vector whose magnitude is zero.
The zero vector may have **any** direction[3] and has the following properties.

Addition of a vector **a** with the zero vector:	$\mathbf{a} + \mathbf{0} \;=\; \mathbf{a}$
Dot product with the zero vector:	$\mathbf{a} \cdot \mathbf{0} \;=\; 0$
Cross product with the zero vector:	$\mathbf{a} \times \mathbf{0} \;=\; \mathbf{0}$

[1] Note: ***Direction*** can be resolved into ***orientation*** and ***sense***. For example, a highway has an orientation (e.g., east-west) and a vehicle traveling east has a sense. Knowing both the orientation of a line and the sense on the line gives direction.

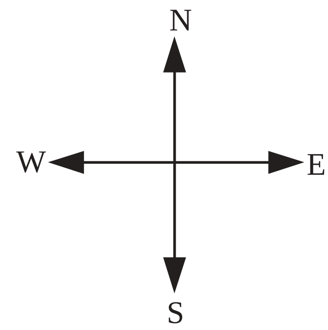
[2] In Autolev, `>` is used to denote a vector, e.g., the vector **v** is represented as `v>` and the ***zero vector*** is `0>`.

[3] Note: It is improper to say the ***zero vector*** has no direction as a vector is **defined** to have both a magnitude and a direction. It is also improper to say the ***zero vector*** has all directions as a vector is defined to have a magnitude and **a** direction (as contrasted with a dyad which has two directions, triad which has three directions, etc.). Thus there are an infinite number of equal zero vectors, each having zero magnitude and **any** direction.

2.4 Unit vectors

A ***unit vector*** is defined as a vector whose magnitude is 1.
Unit vectors are sometimes designated with a special vector hat, e.g., $\hat{\mathbf{u}}$.

Unit vectors are typically introduced as "***sign posts***", e.g., the unit vectors **North**, **South**, **West**, and **East** shown to the right. The direction of unit vectors are chosen to simplify communication and to produce efficient equations. Other useful "sign posts" are[a]



- unit vector directed from one point to another point
- unit vector directed locally vertical
- unit vector tangent to a curve
- unit vector parallel to the edge of an object
- unit vector perpendicular to a surface

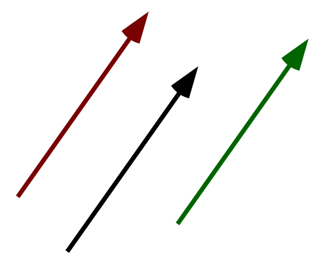
Another way to introduce a unit vector **unitVector** is to define it so it has the same direction as an arbitrary non-zero vector **v** by first determining $|\mathbf{v}|$, the magnitude of **v**, and then writing[b]

$$\mathbf{unitVector} \;=\; \frac{\mathbf{v}}{|\mathbf{v}|} \qquad (1)$$

[a]Homework 2.4 draws various unit vectors.
[b]To avoid divide-by-zero problems during numerical computation, one may write the unit vector in terms of a "small" positive constant ϵ as $\mathbf{unitVector} = \frac{\mathbf{v}}{|\mathbf{v}| + \epsilon}$.

2.5 Equal vectors

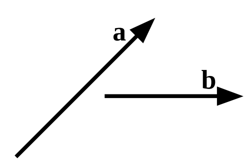
Two vectors are said to be equal (or equivalent) when they have the same magnitude and same direction.[a] The figure to the right shows three ***equal vectors***. Although each vector has a different location, the vectors are equal because they have the same magnitude and direction.[b]



[a]Homework 2.3 draws vectors of different magnitude, ***orientation***, and ***sense***.
[b]Certain vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are ***equal position vectors*** when, in addition to having the same magnitude and direction, the vectors are associated with the same points. Two force vectors are ***equal force vectors*** when the vectors have the same magnitude, direction, and point of application.

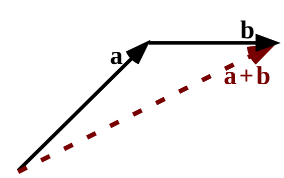
2.6 Vector addition

As graphically shown to the right, adding two vectors $\mathbf{a} + \mathbf{b}$ produces a vector.[a] First, vector **b** is translated[b] so its tail is at the tip of **a**. Next, the vector $\mathbf{a} + \mathbf{b}$ is drawn from the tail of **a** to the tip of the translated **b**.[c]



Properties of vector addition

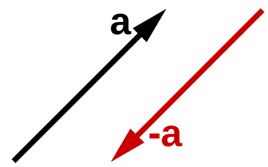
Commutative law:	$\mathbf{a} + \mathbf{b} \;=\; \mathbf{b} + \mathbf{a}$
Associative law:	$(\mathbf{a} + \mathbf{b}) + \mathbf{c} \;=\; \mathbf{a} + (\mathbf{b} + \mathbf{c}) \;=\; \mathbf{a} + \mathbf{b} + \mathbf{c}$
Addition of zero vector:	$\mathbf{a} + \mathbf{0} \;=\; \mathbf{a}$



[a]It does not make sense to add vectors with different units. For example, adding a velocity vector with units of m/sec with an angular velocity vector with units of rad/sec does not produce a vector with sensible units.
[b]Translating **b** does *not* change the magnitude or direction of **b**, and so produces an equal **b**.
[c]Homework 2.7 draws $\mathbf{b} + \mathbf{a}$.

2.7 Vector negation

A graphical representation of negating a vector **a** is shown to the right.[a]

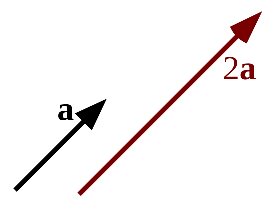


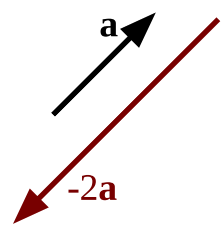
Negating a vector (multiplying the vector by -1) changes the ***sense*** of a vector without changing its magnitude or orientation. In other words, multiplying a vector by -1 reverses the sense of the vector (it points in the opposite direction).

[a]Homework 2.5 draws a vector $-\mathbf{b}$.

2.8 Vector multiplied or divided by a scalar

To the right is a graphical representation of multiplying a vector **a** by a scalar.[a]





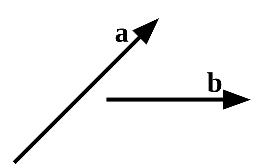
- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the ***sense*** of the vector.
- Dividing a vector **a** by a scalar s_1 is defined as $\frac{\mathbf{a}}{s_1} \triangleq \frac{1}{s_1} * \mathbf{a}$

Properties of multiplication of a vector by a scalar

Commutative law:	$s_1\,\mathbf{a} = \mathbf{a}\,s_1$
Associative law:	$s_1\,(s_2\,\mathbf{a}) = (s_1\,s_2)\,\mathbf{a} = s_2\,(s_1\,\mathbf{a}) = s_1\,s_2\,\mathbf{a}$
Distributive law:	$(s_1 + s_2)\,\mathbf{a} = s_1\,\mathbf{a} + s_2\,\mathbf{a}$
Distributive law:	$s_1\,(\mathbf{a} + \mathbf{b}) = s_1\,\mathbf{a} + s_1\,\mathbf{b}$
Multiplication by one:	$1 * \mathbf{a} = \mathbf{a}$
Multiplication by zero:	$0 * \mathbf{a} = \mathbf{0}$

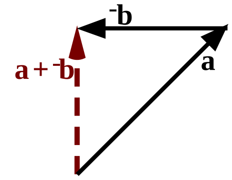
[a]Homework 2.6 multiplies a vector **b** by various scalars.

2.9 Vector subtraction



As graphically shown to the right, the process of subtracting a vector **b** from a vector **a** is simply addition and negation,[a] i.e.,

$$\mathbf{a} - \mathbf{b} \;\triangleq\; \mathbf{a} + -\mathbf{b}$$

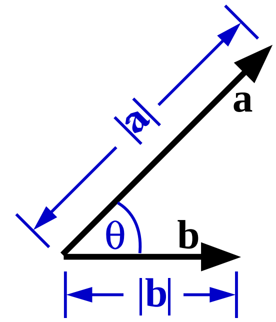


After negating vector **b**, it is translated so the tail of $-\mathbf{b}$ is at the tip of **a**. Next, the vector $\mathbf{a} + -\mathbf{b}$ is drawn from the tail of **a** to the tip of the translated $-\mathbf{b}$.[b]

[a]In most (or all) mathematical processes, subtraction is defined as negation and addition.

[b]Homework 2.8 draws $\mathbf{b} - \mathbf{a} \triangleq \mathbf{b} + -\mathbf{a}$.

2.10 Vector dot product



The ***dot product*** of a vector **a** with a vector **b** is defined in equation (2) in terms of

- $|\mathbf{a}|$ and $|\mathbf{b}|$, the magnitudes of **a** and **b**, respectively
- θ, the smallest angle between **a** and **b** $(0 \leq \theta \leq \pi)$.

$$\mathbf{a} \cdot \mathbf{b} \triangleq |\mathbf{a}| \; |\mathbf{b}| \; \cos(\theta) \qquad (2)$$

When **b** is a unit vector, $|\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b}$ can be interpreted as the ***projection*** of **a** on **b**. Similarly, when **a** is a unit vector, $\mathbf{a} \cdot \mathbf{b}$ can be interpreted as the projection of **b** on **a**.

Rearranging equation (2) produces an expression which is useful for finding the angle between two vectors, i.e., $\theta \underset{(2)}{=} \text{acos}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \; |\mathbf{b}|}\right)$

The dot product is useful for calculating the magnitude of a vector **v**. In view of equation (2), $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$. Hence, one way to determine $|\mathbf{v}|$ is the important relationship in equation (3).

$$\begin{aligned} |\mathbf{v}|^2 &= \mathbf{v} \cdot \mathbf{v} \\ |\mathbf{v}| &= +\sqrt{\mathbf{v} \cdot \mathbf{v}} \end{aligned} \qquad (3)$$

2.10.1 Properties of the dot-product

Dot product with the zero vector	$\mathbf{a} \cdot \mathbf{0} = 0$
Dot product of perpendicular vectors	$\mathbf{a} \cdot \mathbf{b} = 0 \quad$ if $\mathbf{a} \perp \mathbf{b}$
Dot product of vectors having the same direction	$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \; |\mathbf{b}| \quad$ if $\mathbf{a} \parallel \mathbf{b}$
Dot product with vectors scaled by s_1 and s_2	$s_1\mathbf{a} \cdot s_2\mathbf{b} = s_1 \, s_2 \, (\mathbf{a} \cdot \mathbf{b})$
Commutative law	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
Distributive law	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
Distributive law	$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$

2.10.2 Uses for the dot-product

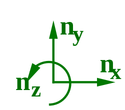
Several uses for the dot-product in geometry, statics, and motion analysis, include

- Calculating an **angle** between two vectors (very useful in geometry)
- Calculating a vector's **magnitude** (e.g., **distance** is the magnitude of a position vector)
- Calculating a **unit vector** in the direction of a vector [as shown in equation (1)]
- Determining when two vectors are **perpendicular**
- Determining the **component** (or measure) of a vector in a certain direction
- Changing a **vector equation** into a **scalar equation** (see Homework 2.19)
- ***Vector exponentiation***

  The definition of ***vector exponentation*** of $\mathbf{v}^n$ for the vector **v** raised to the scalar power n and the specific case of $\mathbf{v}^2$ are shown to the right. Note: $\mathbf{v}^n$ produces a **non-negative** scalar.

$$\begin{aligned} \mathbf{v}^n &\triangleq |\mathbf{v}|^n \underset{(3)}{=} +(\mathbf{v} \cdot \mathbf{v})^{\frac{n}{2}} \\ \mathbf{v}^2 &\triangleq |\mathbf{v}|^2 \underset{(3)}{=} \mathbf{v} \cdot \mathbf{v} \end{aligned} \qquad (4)$$

2.10.3 Special case: Dot-products with orthogonal unit vectors

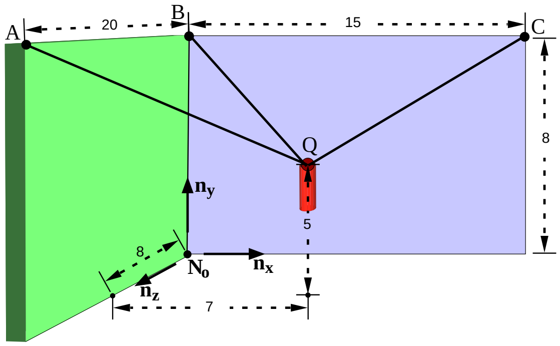


When $\mathbf{n}_\text{x}$, $\mathbf{n}_\text{y}$, $\mathbf{n}_\text{z}$ are ***orthogonal unit*** vectors, it can be shown (see Homework 2.6)

$$(a_x \, \mathbf{n}_\text{x} + a_y \, \mathbf{n}_\text{y} + a_z \, \mathbf{n}_\text{z}) \cdot (b_x \, \mathbf{n}_\text{x} + b_y \, \mathbf{n}_\text{y} + b_z \, \mathbf{n}_\text{z}) = a_x \, b_x + a_y \, b_y + a_z \, b_z$$

2.10.4 Example: Microphone cable lengths and angles (with orthogonal walls)

A microphone Q is attached to three pegs A, B, and C by three cables. Knowing the peg locations and microphone location from point N_o, determine L_A (the length of the cable joining A and Q) and the angle ϕ between line $\overline{AQ}$ and line $\overline{AB}$.



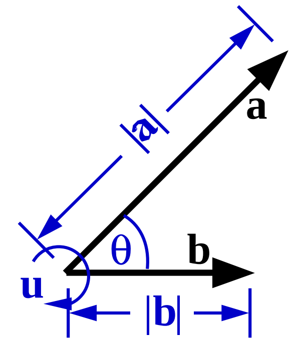
Quantity	Value
Distance from A to B	20 m
Distance from B to C	15 m
Distance from N_o to B	8 m
Q's measure from N_o along back-wall	7 m
Q's height above N_o	5 m
Q's measure from N_o along left-wall	8 m

$$\mathbf{r}^{Q/N_o} = 7\,\mathbf{n}_\mathrm{x} + 5\,\mathbf{n}_\mathrm{y} + 8\,\mathbf{n}_\mathrm{z}$$

Solution:

- Form A's position vector from N_o (inspection): $\mathbf{r}^{A/N_o} = 8\,\mathbf{n}_\mathrm{y} + 20\,\mathbf{n}_\mathrm{z}$
- Form Q's position vector from A (vector addition and rearrangement):
$$\mathbf{r}^{Q/A} = \mathbf{r}^{Q/N_o} - \mathbf{r}^{A/N_o} = 7\,\mathbf{n}_\mathrm{x} + {}^{-}3\,\mathbf{n}_\mathrm{y} + {}^{-}12\,\mathbf{n}_\mathrm{z}$$
- Calculate $\mathbf{r}^{Q/A} \cdot \mathbf{r}^{Q/A}$: $(7\,\mathbf{n}_\mathrm{x} + {}^{-}3\,\mathbf{n}_\mathrm{y} + {}^{-}12\,\mathbf{n}_\mathrm{z}) \cdot (7\,\mathbf{n}_\mathrm{x} + {}^{-}3\,\mathbf{n}_\mathrm{y} + {}^{-}12\,\mathbf{n}_\mathrm{z}) = 202$
- Form L_A, the magnitude of Q's position vector from A: $L_A = \sqrt{\mathbf{r}^{Q/A} \cdot \mathbf{r}^{Q/A}} = \sqrt{202} = 14.2$
- The determination of the angle ϕ starts with the definition of the following dot-product
$$\mathbf{r}^{Q/A} \cdot \mathbf{r}^{B/A} \triangleq \left|\mathbf{r}^{Q/A}\right| \left|\mathbf{r}^{B/A}\right| \cos(\phi)$$
- Subsequent rearrangement and substitution of the known quantities gives
$$\cos(\phi) = \frac{\mathbf{r}^{Q/A} \cdot \mathbf{r}^{B/A}}{\left|\mathbf{r}^{Q/A}\right| \left|\mathbf{r}^{B/A}\right|} = \frac{\mathbf{r}^{Q/A} \cdot \mathbf{r}^{B/A}}{20\,L_A} = \frac{(7\,\mathbf{n}_\mathrm{x} + {}^{-}3\,\mathbf{n}_\mathrm{y} + {}^{-}12\,\mathbf{n}_\mathrm{z}) \cdot ({}^{-}20\,\mathbf{n}_\mathrm{z})}{20 * 14.2} = \frac{240}{284}$$
- Solving for the angle gives $\phi = \operatorname{acos}\left(\frac{240}{284}\right) = 32.32°$.

2.11 Vector cross product



The ***cross product*** of a vector $\mathbf{a}$ with a vector $\mathbf{b}$ is defined in equation (5) in terms of

- $|\mathbf{a}|$ and $|\mathbf{b}|$, the magnitudes of $\mathbf{a}$ and $\mathbf{b}$, respectively
- θ, the smallest angle between $\mathbf{a}$ and $\mathbf{b}$ $(0 \le \theta \le \pi)$.
- $\mathbf{u}$, the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ whose direction is determined by the ***right-hand rule.***[a]

  Note: The coefficient of $\mathbf{u}$ in equation (5) is inherently a ***non-negative*** quantity since $\sin(\theta) \ge 0$ because $0 \le \theta \le \pi$. Hence, $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}| \sin(\theta)$.

[a] The right-hand rule is a recently accepted universal convention, much like driving on the right-hand side of the road in North America. Until 1965, the Soviet Union used the left-hand rule, logically reasoning that the left-hand rule is more convenient because a right-handed person can simultaneously write while performing cross products.

$$\mathbf{a} \times \mathbf{b} \triangleq |\mathbf{a}|\ |\mathbf{b}| \sin(\theta)\ \mathbf{u} \qquad (5)$$

2.11.1 Properties of the cross-product

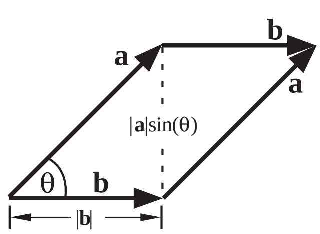
Cross product with the zero vector	$\mathbf{a} \times \mathbf{0} = \mathbf{0}$
Cross product of a vector with itself	$\mathbf{a} \times \mathbf{a} = \mathbf{0}$
Cross product of parallel vectors	$\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if $\mathbf{a} \parallel \mathbf{b}$
Cross product with vectors scaled by s_1 and s_2	$s_1\mathbf{a} \times s_2\mathbf{b} = s_1\, s_2\, (\mathbf{a} \times \mathbf{b})$
Cross products are ***not*** commutative	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
Distributive law	$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
Distributive law	$(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d}$
Cross products are not associative	$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \ne (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
Vector triple cross product	$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$
When $\mathbf{b}$ is a unit vector,	$\lvert\mathbf{a} \times \mathbf{b}\rvert^2 = \mathbf{a} \cdot \mathbf{a} - (\mathbf{a} \cdot \mathbf{b})^2$

A mnemonic for $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$ is "**bac**k **cab**" - as in were you born in the back of a cab?
Many proofs of this formula resolve $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ into orthogonal unit vectors (e.g., $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$) and equate components.

2.11.2 Uses for the cross-product

Several uses for the cross-product in geometry, statics, and motion analysis, include

- Calculating **perpendicular** vectors, e.g., $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- Determining when two vectors are **parallel**, e.g., $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ when $\mathbf{a}$ is parallel to $\mathbf{b}$
- Calculating the **moment** of a force or linear momentum, e.g., $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ and $\mathbf{H} = \mathbf{r} \times m\,\mathbf{v}$
- Calculating **velocity/acceleration** formulas, e.g., $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ and $\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$
- Calculating the **area of a triangle** whose sides have length $|\mathbf{a}|$ and $|\mathbf{b}|$



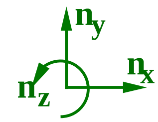
The ***area of a triangle*** Δ is half the area of a parallelogram.[a] Since one geometrical interpretation of $|\mathbf{a} \times \mathbf{b}|$ is the ***area of a parallelogram*** having sides of length $|\mathbf{a}|$ and $|\mathbf{b}|$,

$$\Delta = \frac{1}{2}\ |\mathbf{a} \times \mathbf{b}| \qquad (6)$$

[a] Homework 2.11 shows the utility of equation (6) for ***surveying***.

2.11.3 Special case: Cross-products with right-handed, orthogonal, unit vectors

Given **right-handed orthogonal unit** vectors $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$ and two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ that are expressed in terms of $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$ as shown to the right, calculating $\mathbf{a} \times \mathbf{b}$ with the distributive property of the cross product happens to be equal to the ***determinant*** of the matrix and the expression shown below.[a]



$$\mathbf{a} = a_x\,\mathbf{n}_x + a_y\,\mathbf{n}_y + a_z\,\mathbf{n}_z$$
$$\mathbf{b} = b_x\,\mathbf{n}_x + b_y\,\mathbf{n}_y + b_z\,\mathbf{n}_z$$

[a] This is proved in Homework 2.9.

$$\mathbf{a} \times \mathbf{b} \;=\; \det \begin{bmatrix} \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \;=\; (a_y\,b_z - a_z\,b_y)\,\mathbf{n}_x \;+\; (a_z\,b_x - a_x\,b_z)\,\mathbf{n}_y \;+\; (a_x\,b_y - a_y\,b_x)\,\mathbf{n}_z$$

2.12 Scalar triple product and the volume of a tetrahedron

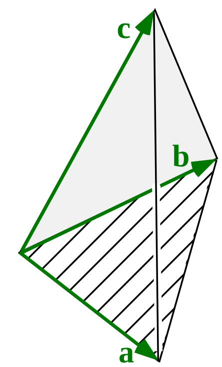
The ***scalar triple product*** of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the scalar defined in the various ways shown in equation (7).[4] Homework 2.13 shows how ***determinants*** can calculate certain scalar triple products.

$$\text{ScalarTripleProduct} \;\triangleq\; \boxed{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \;=\; \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}} \;=\; \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} \;=\; \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} \qquad (7)$$

A geometrical interpretation of $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ is the ***volume of a parallelepiped*** having sides of length $|\mathbf{a}|$, $|\mathbf{b}|$, and $|\mathbf{c}|$. The formula for the ***volume of a tetrahedron*** whose sides are described by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is

$$\text{Tetrahedron Volume} \;=\; \frac{1}{6}\;\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

This formula is useful for volume (***surveying*** cut and fill) calculations as well as 3D (***CAD***) solid geometry mass property calculations.



2.13 Dyads and dyadics (vector multiplied by a vector)

The ***dyad*** $\underline{\mathbf{D}}$ that results from the multiplication of the vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as $\underline{\mathbf{D}} \triangleq \mathbf{a} * \mathbf{b} = \mathbf{a}\,\mathbf{b}$. To clearly distinguish a dyad from a scalar or vector, it is common to use two arrows, two hats, two tildes, or two underlines, i.e., $\vec{\vec{\mathrm{D}}}$, $\hat{\hat{\mathrm{D}}}$, $\tilde{\tilde{\mathrm{D}}}$, or $\underline{\underline{\mathrm{D}}}$, or to use a single line under a bold-face symbol, i.e., $\underline{\mathbf{D}}$.[5]

The **sum** of two or more dyads is called a ***dyadic***. For example, letting $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be vectors, the dyads $\mathbf{ab}$ and $\mathbf{cd}$ can be formed. Their sum $\mathbf{ab} + \mathbf{cd}$ is a dyadic.[6]

2.13.1 The zero dyadic and the unit dyadic

The ***zero dyadic*** $\underline{\mathbf{0}}$ is defined as the dyad with two zero vectors.

$$\boxed{\underline{\mathbf{0}} \triangleq \mathbf{0} * \mathbf{0}} \qquad (8)$$

[4] Although parentheses make equation (7) clearer, i.e., ScalarTripleProduct $\triangleq \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, the parentheses are unnecessary because the cross product $\mathbf{b} \times \mathbf{c}$ ***must*** be performed before the dot product for a sensible result to be produced.

[5] In Autolev, `>>` is used to denote a dyadic, e.g., the dyadic $\underline{\mathbf{D}}$ is represented as `D>>` and the unit dyadic is `1>>`.

[6] In general, the ***dyadic*** $\mathbf{ab} + \mathbf{cd}$ is not a ***dyad*** because it cannot be written as a vector multiplied by a vector. Dyadics differ from vectors in that the **sum** of two vectors is a vector whereas the **sum** of two dyads is not necessarily a dyad.

The ***unit dyadic*** is denoted $\underline{\mathbf{1}}$ and is defined by its property in equation (9).[a] As shown in Section 2.14, the unit dyadic can be written in terms of the orthogonal unit vectors $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$ as shown in equation (10).[b]

$$\boxed{\underline{\mathbf{1}} \cdot \mathbf{v} \;=\; \mathbf{v} \cdot \underline{\mathbf{1}} \;=\; \mathbf{v} \quad \text{where } \mathbf{v} \text{ is } \textit{any} \text{ vector.}} \tag{9}$$

$$\boxed{\underline{\mathbf{1}} \;=\; \mathbf{b}_x\,\mathbf{b}_x \;+\; \mathbf{b}_y\,\mathbf{b}_y \;+\; \mathbf{b}_z\,\mathbf{b}_z} \tag{10}$$

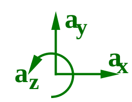
[a] The unit dyadic is similar to the identity matrix I whose defining property is $I * x \;=\; x * I \;=\; x$ where x is ***any*** matrix.

[b] The orthogonal unit vectors $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$ do ***not*** have to be right-handed for the unit dyadic to be expressed as $\underline{\mathbf{1}} = \mathbf{b}_x\,\mathbf{b}_x + \mathbf{b}_y\,\mathbf{b}_y + \mathbf{b}_z\,\mathbf{b}_z$. Unless a cross-product is involved, the right-handed nature of the vectors is irrelevant.

2.13.2 Properties of dyadics associated with the vectors a, b, c, d, and w

Dyads are **not** commutative:	$\mathbf{ab} \neq \mathbf{ba}$
Distributive law:	$\mathbf{a}\,(\mathbf{b}+\mathbf{c}) \;=\; \mathbf{ab} \;+\; \mathbf{ac}$
Distributive law:	$(\mathbf{a}+\mathbf{b})\,(\mathbf{c}+\mathbf{d}) \;=\; \mathbf{ac} \;+\; \mathbf{ad} \;+\; \mathbf{bc} \;+\; \mathbf{bd}$
Pre-dot product:	$\mathbf{w} \cdot (\mathbf{ab}+\mathbf{cd}) \;=\; (\mathbf{w}\cdot\mathbf{a})\,\mathbf{b} \;+\; (\mathbf{w}\cdot\mathbf{c})\,\mathbf{d}$
Post-dot product:	$(\mathbf{ab}+\mathbf{cd}) \cdot \mathbf{w} \;=\; \mathbf{a}\,(\mathbf{b}\cdot\mathbf{w}) \;+\; \mathbf{c}\,(\mathbf{d}\cdot\mathbf{w})$
Pre-cross product:	$\mathbf{w} \times (\mathbf{ab}+\mathbf{cd}) \;=\; (\mathbf{w}\times\mathbf{a})\,\mathbf{b} \;+\; (\mathbf{w}\times\mathbf{c})\,\mathbf{d}$
Post-cross product:	$(\mathbf{ab}+\mathbf{cd}) \times \mathbf{w} \;=\; \mathbf{a}\,(\mathbf{b}\times\mathbf{w}) \;+\; \mathbf{c}\,(\mathbf{d}\times\mathbf{w})$
Vector multiplication:	$s_1\mathbf{a} * s_2\mathbf{b} \;=\; s_1\,s_2\,\mathbf{a}\,\mathbf{b}$ (s_1 and s_2 are scalars)
Dot product with **0**:	$\underline{\mathbf{D}} \cdot \mathbf{0} \;=\; \mathbf{0}$ ($\underline{\mathbf{D}}$ is any dyadic and $\mathbf{0}$ is the zero vector)

2.13.3 Dyadic examples



Complete the following calculations using the orthogonal unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$.

$$\underline{\mathbf{0}} \cdot (\mathbf{a}_x + 5\,\mathbf{a}_y) \;=\; \underline{\qquad}$$

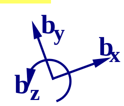
$$\underline{\mathbf{1}} \cdot (\mathbf{a}_x + 5\,\mathbf{a}_y) \;=\; \underline{\qquad\qquad}$$

$$\underline{\mathbf{1}} \;=\; \mathbf{a}_x{*}\mathbf{a}_x \;+\; \mathbf{a}_y{*}\mathbf{a}_y \;+\; \underline{\qquad\quad}$$

$$(\mathbf{a}_x \;+\; 2\,\mathbf{a}_y \;+\; 3\,\mathbf{a}_z) * (4\,\mathbf{a}_x \;+\; 5\,\mathbf{a}_y \;+\; 6\,\mathbf{a}_z) \;=\; 4\,\mathbf{a}_x\,\mathbf{a}_x \;+\; 5\,\mathbf{a}_x\,\mathbf{a}_y \;+\; \underline{\qquad} \;+\; \underline{\qquad} \;+\; \underline{\qquad} \;+\; \underline{\qquad} \;+\; \underline{\qquad} \;+\; \underline{\qquad} \;+\; 18\,\mathbf{a}_z\,\mathbf{a}_z$$

$$(\mathbf{a}_y\,\mathbf{a}_z \;+\; 4\,\mathbf{a}_z\,\mathbf{a}_x \;+\; 5\,\mathbf{a}_z\,\mathbf{a}_y) \cdot (\mathbf{a}_x \;+\; 2\,\mathbf{a}_y \;+\; 3\,\mathbf{a}_z) \;=\; 3\,\mathbf{a}_y \;+\; \underline{\qquad}$$

$$(\mathbf{a}_x \;+\; 2\,\mathbf{a}_y \;+\; 3\,\mathbf{a}_z) \cdot (\mathbf{a}_y\,\mathbf{a}_z \;+\; 4\,\mathbf{a}_z\,\mathbf{a}_x \;+\; 5\,\mathbf{a}_z\,\mathbf{a}_y) \;=\; 2\,\mathbf{a}_z \;+\; \underline{\qquad} \;+\; \underline{\qquad}$$



Complete the next set of calculations with the orthogonal unit vectors $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$.

$$(\mathbf{a}_x + 2\,\mathbf{a}_y) * (\mathbf{b}_x + 3\,\mathbf{b}_z) \;=\; \mathbf{a}_x\,\mathbf{b}_x \;+\; 3\,\mathbf{a}_x\,\mathbf{b}_z \;+\; 2\,\mathbf{a}_y\,\mathbf{b}_x \;+\; \underline{\qquad}$$

$$(\mathbf{a}_x\,\mathbf{b}_x \;+\; 3\,\mathbf{a}_x\,\mathbf{b}_z \;+\; 2\,\mathbf{a}_y\,\mathbf{b}_x \;+\; 6\,\mathbf{a}_y\,\mathbf{b}_z) \cdot \mathbf{b}_x \;=\; \mathbf{a}_x \;+\; \underline{\qquad}$$

$$\mathbf{a}_y \cdot (\mathbf{a}_x\,\mathbf{b}_x \;+\; 3\,\mathbf{a}_x\,\mathbf{b}_z \;+\; 2\,\mathbf{a}_y\,\mathbf{b}_x \;+\; 6\,\mathbf{a}_y\,\mathbf{b}_z) \;=\; 2\,\mathbf{b}_x \;+\; \underline{\qquad}$$

2.14 Optional**: Unit dyadic expressed with orthogonal unit vectors

To prove $\underline{\mathbf{1}} \;=\; \mathbf{b}_x\,\mathbf{b}_x + \mathbf{b}_y\,\mathbf{b}_y + \mathbf{b}_z\,\mathbf{b}_z$ in equation (10), note that equations (3.1) and (3.4) allow an arbitrary vector $\mathbf{v}$ to be expressed in terms of ***any*** orthogonal unit vectors $\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$ as

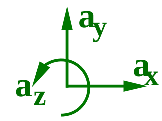
$$\mathbf{v} \;=\; (\mathbf{v}\cdot\mathbf{b}_x)\,\mathbf{b}_x \;+\; (\mathbf{v}\cdot\mathbf{b}_y)\,\mathbf{b}_y \;+\; (\mathbf{v}\cdot\mathbf{b}_z)\,\mathbf{b}_z \;=\; \mathbf{v} \cdot (\mathbf{b}_x\,\mathbf{b}_x \;+\; \mathbf{b}_y\,\mathbf{b}_y \;+\; \mathbf{b}_z\,\mathbf{b}_z)$$

As described in Section 2.13, the unit dyadic is defined by its property $\mathbf{v} \cdot \underline{\mathbf{1}} = \mathbf{v}$, hence

$$\mathbf{v} \cdot \underline{\mathbf{1}} \;=\; \mathbf{v} \cdot (\mathbf{b}_x\,\mathbf{b}_x + \mathbf{b}_y\,\mathbf{b}_y + \mathbf{b}_z\,\mathbf{b}_z) \qquad \text{or} \qquad \mathbf{v} \cdot [\underline{\mathbf{1}} - (\mathbf{b}_x\,\mathbf{b}_x + \mathbf{b}_y\,\mathbf{b}_y + \mathbf{b}_z\,\mathbf{b}_z)] \;=\; 0$$

The proof is completed by noting that $\mathbf{v}$ is an **arbitrary vector** (e.g., not-necessarily $\mathbf{0}$).

2.15 Vector operations with Autolev



Vector operations such as addition, scalar multiplication, dot-products, and cross-products can be performed with Autolev as shown below.

```
   (1) % File: VectorDemonstration.al         % The percent sign denotes a comment
   (2) RigidFrame A                           % Create orthogonal unit vectors Ax>, Ay>, Az>
   (3) V> = Vector( A, 2, 3, 4 )              % Construct a vector V>
-> (4) V> = 2*Ax> + 3*Ay> + 4*Az>

   (5) W> = Vector( A, 6, 7, 8 )              % Construct a vector W>
-> (6) W> = 6*Ax> + 7*Ay> + 8*Az>

   (7) V5> = 5 * V>                           % Multiply V> by 5
-> (8) V5> = 10*Ax> + 15*Ay> + 20*Az>

   (9) magV = GetMagnitude( V> )              % Magnitude of V>
-> (10) magV = 5.385165

   (11) unitV> = GetUnitVector( V> )           % Unit vector in the direction of V>
-> (12) unitV> = 0.3713907*Ax> + 0.557086*Ay> + 0.7427814*Az>

   (13) addVW> = V> + W>                       % Add vectors V> and W>
-> (14) addVW> = 8*Ax> + 10*Ay> + 12*Az>

   (15) dotVW = Dot( V>, W> )                  % Dot product of V> and W>
-> (16) dotVW = 65

   (17) crossVW> = Cross( V>, W> )             % Cross product of V> and W>
-> (18) crossVW> = -4*Ax> + 8*Ay> - 4*Az>

   (19) crossWWV> = Cross( W>, Cross(W>,V>) )  % Vector triple cross product
-> (20) crossWWV> = 92*Ax> + 8*Ay> - 76*Az>

   (21) multVW>> = V> * W>                     % Form a dyadic by multiplying V> and W>
-> (22) multVW>> = 12*Ax>*Ax> + 14*Ax>*Ay> + 16*Ax>*Az> + 18*Ay>*Ax> + 21*Ay>*
        Ay> + 24*Ay>*Az> + 24*Az>*Ax> + 28*Az>*Ay> + 32*Az>*Az>

   (23) dotVWithZeroVector  = Dot( V>, 0>  )   % Dot product of V> with the zero vector
-> (24) dotVWithZeroVector = 0

   (25) dotVWithUnitDyadic> = Dot( V>, 1>> )   % Dot product of V> with the unit dyadic
-> (26) dotVWithUnitDyadic> = 2*Ax> + 3*Ay> + 4*Az>
```